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Differentiation

DERIVATIVE OF A FUNCTION

Let $y = f(x)$ be a given continuous function. Then, the value of y depends upon the value of x and it changes with a change in the value of x. We use the word increment to denote a small change, i.e., an increase or decrease, in the values of x and y.

Let δy be an increment in y, corresponding to an increment δx in x. Then,

$$y = f(x) \text{ and } y + \delta y = f(x + \delta x).$$

On subtraction, we get $\delta y = f(x + \delta x) - f(x)$.

$$\therefore \quad \frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}.$$

$$\text{So,} \quad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

The above limit, if it exists finitely, is called the *derivative* or *differential coefficient* of $y = f(x)$ with respect to x, and it is denoted by

$$\frac{dy}{dx} \text{ or } \frac{d}{dx}\{f(x)\} \text{ or } f'(x).$$

The process of finding the derivative is known as differentiation.

REMARK $\quad \dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim_{\delta x \to 0} \dfrac{f(x + \delta x) - f(x)}{\delta x}.$

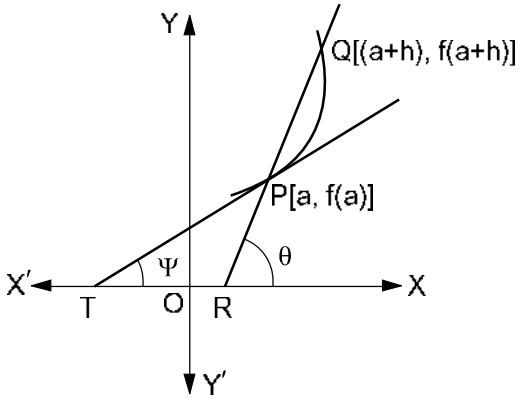
DERIVATIVE AT A POINT *The value of $f'(x)$, obtained by putting $x = a$, is called the derivative of $f(x)$ at $x = a$, and it is denoted by $f'(a)$ or $\left\{\dfrac{dy}{dx}\right\}_{x=a}$.*

If $f'(a)$ exists, we say that $f(x)$ is *differentiable* at $x = a$.

If $f'(x)$ exists for every value of x in the domain of the function, we say that $f(x)$ is differentiable.

GEOMETRICAL SIGNIFICANCE OF A DERIVATIVE

Let $y = f(x)$ be a continuous function. Then, we draw its graph. Suppose it is a curve. Let a be a point in the domain of the given function. Let $P[a, f(a)]$ be a point on this curve, and let $Q[a + h, f(a + h)]$ be some neighbouring point on it.



Slope of chord $PQ = \dfrac{f(a + h) - f(a)}{(a + h - a)}$

$$= \frac{f(a+h) - f(a)}{h}.$$

Let the point Q move along the curve such that $Q \to P$.

As Q comes closer and closer to P along the curve, the chord QP approaches the tangent at P.

This happens when $h \to 0$.

$\therefore \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{Q \to P}$ (slope of chord PQ)

or $\quad f'(a) =$ the slope of the tangent at P.

Hence, *the derivative of $f(x)$ at $x = a$ is the slope of the tangent to the curve $y = f(x)$ at the point $[a, f(a)]$.*

PHYSICAL SIGNIFICANCE OF A DERIVATIVE Let $s = f(t)$ be a function representing the distance travelled by a particle moving in a straight line in time t.

Let $(s + \delta s)$ be the distance travelled by it in time $(t + \delta t)$.

$\therefore \quad s + \delta s = f(t + \delta t).$

On subtraction, we get $\delta s = f(t + \delta t) - f(t)$.

$\therefore \quad$ average velocity $= \frac{\delta s}{\delta t} = \frac{f(t+\delta t) - f(t)}{\delta t}.$

Now, when $\delta t \to 0$, the average velocity becomes the instantaneous velocity.

$\therefore$ velocity at time $t = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} = f'(t).$

DIFFERENTIATION FROM THE FIRST PRINCIPLE Obtaining the derivative of a given function by using the definition is called *differentiation from the first principle* or *ab initio* or *by delta method.*

Some Important Derivatives Using the First Principle

THEOREM 1 *From the first principle, we have*

$$\frac{d}{dx}(x^n) = nx^{n-1},$$ *where n is a fixed number, integer or rational.*

PROOF Let $y = x^n$. … (i)

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^n$. … (ii)

On subtracting (i) from (ii), we get $\delta y = (x + \delta x)^n - x^n$

or $\frac{\delta y}{\delta x} = \frac{(x+\delta x)^n - x^n}{\delta x}.$

$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{(x+\delta x)\to x} \frac{(x+\delta x)^n - x^n}{(x+\delta x) - x} \quad [\because\ \delta x \to 0 \text{ means } (x+\delta x) \to x]$$

$$= nx^{n-1} \qquad [\because\ \lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1}].$$

Thus, $\dfrac{dy}{dx} = nx^{n-1}$, i.e., $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

EXAMPLE *Find the derivatives of*

(i) x^9 *(ii)* x^{-3} *(iii)* $\sqrt[3]{x}$ *(iv)* $\dfrac{1}{\sqrt{x}}$

SOLUTION We know that $\dfrac{d}{dx}(x^n) = nx^{n-1}$. So, we have

(i) $\dfrac{d}{dx}(x^9) = 9 \cdot x^{(9-1)} = 9x^8$.

(ii) $\dfrac{d}{dx}(x^{-3}) = (-3) \cdot x^{(-3-1)} = -3x^{-4} = \dfrac{-3}{x^4}$.

(iii) $\dfrac{d}{dx}(\sqrt[3]{x}) = \dfrac{d}{dx}(x^{1/3}) = \dfrac{1}{3}x^{\left(\frac{1}{3}-1\right)} = \dfrac{1}{3}x^{-2/3}$.

(iv) $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = \dfrac{d}{dx}(x^{-1/2}) = -\dfrac{1}{2} \cdot x^{\left(-\frac{1}{2}-1\right)} = -\dfrac{1}{2}x^{-3/2}$.

THEOREM 2 *From the first principle, we have* $\dfrac{d}{dx}(e^x) = e^x$.

PROOF Let $y = e^x$ and $y + \delta y = e^{x+\delta x}$. Then, $\dfrac{\delta y}{\delta x} = \dfrac{e^{(x+\delta x)} - e^x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{e^{(x+\delta x)} - e^x}{\delta x} = \lim_{\delta x\to 0} e^x \left(\frac{e^{\delta x} - 1}{\delta x}\right)$$

$$= e^x \cdot \lim_{\delta x\to 0} \left(\frac{e^{\delta x} - 1}{\delta x}\right) = e^x \qquad \left[\because\ \lim_{\delta x\to 0} \frac{e^{\delta x} - 1}{\delta x} = 1\right].$$

Thus, $\dfrac{dy}{dx} = e^x$, i.e., $\dfrac{d}{dx}(e^x) = e^x$.

THEOREM 3 *From the first principle, we have* $\dfrac{d}{dx}(\sin x) = \cos x$.

PROOF Let $y = \sin x$ and $y + \delta y = \sin(x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{\sin(x+\delta x) - \sin x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\left[\because \ \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\left(\frac{\delta x}{2}\right) \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} = (\cos x \times 1) = \cos x.$$

$$\therefore \ \frac{dy}{dx} = \cos x, \quad \text{i.e.,} \quad \frac{d}{dx}(\sin x) = \cos x.$$

THEOREM 4 *From the first principle, we have* $\frac{d}{dx}(\cos x) = -\sin x.$

PROOF Let $y = \cos x$ and $y + \delta y = \cos(x + \delta x).$

Then, $\frac{\delta y}{\delta x} = \frac{\cos(x + \delta x) - \cos x}{\delta x}.$

$$\therefore \ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cos(x + \delta x) - \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right) \cdot \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\left[\because \ \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)} = (-\sin x) \times 1 = -\sin x.$$

$$\therefore \ \frac{dy}{dx} = -\sin x, \quad \text{i.e.,} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

THEOREM 5 *From the first principle, we have* $\frac{d}{dx}(\tan x) = \sec^2 x.$

PROOF Let $y = \tan x$ and $y + \delta y = \tan(x + \delta x).$

Then, $\frac{\delta y}{\delta x} = \frac{\tan(x + \delta x) - \tan x}{\delta x}.$

$$\therefore \ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\tan(x+\delta x) - \tan x}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\delta x \cdot \cos(x+\delta x)\cdot \cos x}$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x \cdot \cos(x+\delta x)}$$

$$[\because \ \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x+\delta x)}$$

$$= \left(\frac{1}{\cos x} \times 1 \times \frac{1}{\cos x}\right) = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x, \quad \text{i.e.,} \quad \frac{d}{dx}(\tan x) = \sec^2 x.$$

THEOREM 6 *From the first principle, we have* $\frac{d}{dx}(\sec x) = \sec x \ \tan x.$

PROOF Let $y = \sec x$ and $y + \delta y = \sec(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\sec(x+\delta x) - \sec x}{\delta x}$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sec(x+\delta x) - \sec x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{1}{\cos(x+\delta x)} - \frac{1}{\cos x}\right\}}{\delta x} = \lim_{\delta x \to 0} \frac{\cos x - \cos(x+\delta x)}{\delta x \cdot \cos(x+\delta x)\cdot \cos x}$$

$$= \lim_{\delta x \to 0} \frac{2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cdot \cos x \cdot \delta x}$$

$$\left[\because \ \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)\right]$$

$$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x+\delta x)} \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}$$

$$= \left(\frac{1}{\cos x} \times \sin x \times \frac{1}{\cos x} \times 1\right) = \sec x \ \tan x.$$

$$\therefore \quad \frac{dy}{dx} = \sec x \ \tan x, \quad \text{i.e.,} \quad \frac{d}{dx}(\sec x) = \sec x \ \tan x.$$

THEOREM 7 *From the first principle, we have* $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \ \cot x.$

PROOF Let $y = \operatorname{cosec} x$ and $y + \delta y = \operatorname{cosec}(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\operatorname{cosec}(x+\delta x) - \operatorname{cosec} x}{\delta x}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\operatorname{cosec}(x+\delta x) - \operatorname{cosec} x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{1}{\sin(x+\delta x)} - \frac{1}{\sin x}\right\}}{\delta x} = \lim_{\delta x \to 0} \frac{\sin x - \sin(x+\delta x)}{\sin(x+\delta x)\cdot \sin x \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\cos\left\{x + \frac{\delta x}{2}\right\} \cdot \sin\frac{\delta x}{2}}{\sin(x+\delta x)\cdot \sin x \cdot \delta x}$$

$$\left[\because \quad \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\frac{1}{\sin x} \cdot \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)} \cdot \lim_{\delta x \to 0} \frac{1}{\sin(x+\delta x)}$$

$$= \left(-\frac{1}{\sin x} \times \cos x \times 1 \times \frac{1}{\sin x}\right) = -\operatorname{cosec} x \cot x.$$

$$\therefore \quad \frac{dy}{dx} = -\operatorname{cosec} x \cot x, \quad \text{i.e.,} \quad \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$$

THEOREM 8 *From the first principle, we have* $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

PROOF Let $y = \cot x$ and $y + \delta y = \cot(x + \delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\cot(x+\delta x) - \cot x}{\delta x}.$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\cot(x+\delta x) - \cot x}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\frac{\cos(x+\delta x)}{\sin(x+\delta x)} - \frac{\cos x}{\sin x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin x \cos(x+\delta x) - \cos x \sin(x+\delta x)}{\sin(x+\delta x)\cdot \sin x \cdot \delta x}$$

$$= \frac{1}{\sin x} \cdot \lim_{\delta x \to 0} \frac{-\sin \delta x}{\sin(x+\delta x)\cdot \delta x}$$

$$[\because \quad \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= \frac{-1}{\sin x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \cdot \frac{1}{\lim\limits_{\delta x \to 0} \sin(x+\delta x)}$$

$$= \left\{-\frac{1}{\sin x} \cdot 1 \cdot \frac{1}{\sin x}\right\} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x.$$

Thus, $\frac{dy}{dx} = -\operatorname{cosec}^2 x$, i.e., $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.$

SUMMARY

We may summarise the above results as given below:

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(\sin x) = \cos x$ (iv) $\frac{d}{dx}(\cos x) = -\sin x$

(v) $\frac{d}{dx}(\tan x) = \sec^2 x$ (vi) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(vii) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$ (viii) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$.

Some more Results on Differentiation

THEOREM 9 *The derivative of a constant function is zero, i.e.,* $\frac{d}{dx}(c) = 0$.

PROOF Let $f(x) = c$ be a constant function. Then, for any change δx in x, there is no change in the value of the function.

$\therefore$ $f(x) = c$ and $f(x + \delta x) = c$.

So, from first principle, we have

$$\frac{d}{dx}(c) = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \to 0} \left(\frac{c-c}{\delta x}\right) = 0.$$

Hence, $\frac{d}{dx}(c) = 0$, where c is a constant.

THEOREM 10 *Let $f(x)$ be a differentiable function and let c be a fixed real number. Then, $c \cdot f(x)$ is also differentiable and* $\frac{d}{dx}\{c \cdot f(x)\} = c \cdot \frac{d}{dx}\{f(x)\}$.

PROOF Let $y = c \cdot f(x)$ and $y + \delta y = c \cdot f(x + \delta x)$.

Then, $\delta y = c \cdot f(x + \delta x) - c \cdot f(x)$.

$\therefore$ $\frac{\delta y}{\delta x} = \frac{c \cdot f(x+\delta x) - c \cdot f(x)}{\delta x}$.

So, $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{c \cdot f(x+\delta x) - c \cdot f(x)}{\delta x}$

$$= c \cdot \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = c \cdot \frac{d}{dx}\{f(x)\}.$$

Thus, $\frac{dy}{dx} = c \cdot \frac{d}{dx}\{f(x)\}$, i.e., $\frac{d}{dx}\{c \cdot f(x)\} = c \cdot \frac{d}{dx}\{f(x)\}$.

Now, $f(x)$ being differentiable, it follows that $\frac{d}{dx}\{f(x)\}$ exists and therefore, $c \cdot \frac{d}{dx}\{f(x)\}$ exists.

Consequently, $\frac{d}{dx}\{c\cdot f(x)\}$ exists.

This shows that $c\cdot f(x)$ is differentiable.

EXAMPLE 1 *Find the derivative of*
(i) $8x^3$ *(ii)* $6\sqrt{x}$ *(iii)* $5e^x$ *(iv)* 9×2^x

SOLUTION (i) $\frac{d}{dx}(8x^3) = 8\cdot\frac{d}{dx}(x^3) = 8\times 3x^2 = 24x^2.$

(ii) $\frac{d}{dx}(6\sqrt{x}) = 6\cdot\frac{d}{dx}(x^{1/2}) = 6\cdot\frac{1}{2}x^{-1/2} = \frac{3}{\sqrt{x}}.$

(iii) $\frac{d}{dx}(5e^x) = 5\cdot\frac{d}{dx}(e^x) = 5e^x.$

(iv) $\frac{d}{dx}(9\cdot 2^x) = 9\cdot\frac{d}{dx}(2^x) = 9\times 2^x(\log 2).$

THEOREM 11 *If $f(x)$ and $g(x)$ are differentiable functions then $f(x)+g(x)$ is also differentiable, and*

$$\frac{d}{dx}\{f(x)+g(x)\} = \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}.$$

PROOF Let $y = f(x)+g(x)$ and let $y+\delta y = f(x+\delta x)+g(x+\delta x)$.

Then, $\frac{\delta y}{\delta x} = \frac{\{f(x+\delta x)+g(x+\delta x)\}-\{f(x)+g(x)\}}{\delta x}$

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\{f(x+\delta x)+g(x+\delta x)\}-\{f(x)+g(x)\}}{\delta x}$$

$$= \lim_{\delta x\to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}+\frac{g(x+\delta x)-g(x)}{\delta x}\right\}$$

$$= \lim_{\delta x\to 0}\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\}+\lim_{\delta x\to 0}\left\{\frac{g(x+\delta x)-g(x)}{\delta x}\right\}$$

$$= \frac{d}{dx}\{f(x)\}+\frac{d}{dx}\{g(x)\}.$$

$$\therefore \quad \frac{d}{dx}\{f(x)+g(x)\} = \frac{d}{dx}\{f(x)\}+\frac{d}{dx}\{g(x)\}.$$

Since $f(x)$ and $g(x)$ are differentiable, it follows that $\frac{d}{dx}\{f(x)\}$ as well as $\frac{d}{dx}\{g(x)\}$ exists.

Consequently, $\frac{d}{dx}\{f(x)\}+\frac{d}{dx}\{g(x)\}$ exists.

So, by the above result, $\frac{d}{dx}\{f(x)+g(x)\}$ exists.

Hence, $\{f(x)+g(x)\}$ is differentiable.

REMARK We may extend the above result for a finite number of differentiable functions. Thus, we have

$$\frac{d}{dx}[f_1(x) + f_2(x) + \ldots + f_n(x)] = \frac{d}{dx}\{f_1(x)\} + \frac{d}{dx}\{f_2(x)\} + \ldots + \frac{d}{dx}\{f_n(x)\}.$$

EXAMPLE 2 *Find the derivative of* $(x^3 + e^x + 3^x + \cot x)$ *with respect to* x.

SOLUTION We have

$$\frac{d}{dx}(x^3 + e^x + 3^x + \cot x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(e^x) + \frac{d}{dx}(3^x) + \frac{d}{dx}(\cot x)$$
$$= 3x^2 + e^x + 3^x(\log 3) - \operatorname{cosec}^2 x.$$

EXAMPLE 3 *Find the derivative of* $\left(9x^2 + \frac{3}{x} + 5\sin x\right)$ *with respect to* x.

SOLUTION We have $\frac{d}{dx}(9x^2 + \frac{3}{x} + 5\sin x)$

$$= 9 \cdot \frac{d}{dx}(x^2) + 3 \cdot \frac{d}{dx}(x^{-1}) + 5 \cdot \frac{d}{dx}(\sin x)$$

$$= 9 \times 2x + 3 \cdot (-1)\, x^{-2} + 5\cos x = 18x - \frac{3}{x^2} + 5\cos x.$$

THEOREM 12 *If* $f(x)$ *and* $g(x)$ *be differentiable functions then* $\{f(x) - g(x)\}$ *is also differentiable, and*

$$\frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx}\{f(x)\} - \frac{d}{dx}\{g(x)\}.$$

PROOF We have $\frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx}\{f(x) + (-1) \cdot g(x)\}$

$$= \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{(-1) \cdot g(x)\}$$

$$= \frac{d}{dx}\{f(x)\} + (-1) \cdot \frac{d}{dx}\{g(x)\} = \frac{d}{dx}\{f(x)\} - \frac{d}{dx}\{g(x)\}.$$

$$\therefore \quad \frac{d}{dx}\{f(x) - g(x)\} = \frac{d}{dx}\{f(x)\} - \frac{d}{dx}\{g(x)\}.$$

Now, $f(x)$ and $g(x)$ being differentiable, it follows that $\frac{d}{dx}\{f(x)\}$ and $\frac{d}{dx}\{g(x)\}$ both exist.

Consequently, $\frac{d}{dx}\{f(x)\} - \frac{d}{dx}\{g(x)\}$ exists.

So, by the above result, $\frac{d}{dx}\{f(x) - g(x)\}$ exists.

Hence, $\{f(x) - g(x)\}$ is differentiable.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate the following functions with respect to x:*

$$\left(x^2+\frac{4}{x^2}-\frac{2}{3}\tan x+6e\right)$$

SOLUTION $\dfrac{d}{dx}\left(x^2+\dfrac{4}{x^2}-\dfrac{2}{3}\tan x+6e\right)$

$$=\frac{d}{dx}(x^2)+4\cdot\frac{d}{dx}(x^{-2})-\frac{2}{3}\cdot\frac{d}{dx}(\tan x)+6\cdot\frac{d}{dx}(e)$$

$$=2x+4\cdot(-2)\,x^{-3}-\frac{2}{3}\sec^2 x+6\times 0 \qquad \left[\because\ \frac{d}{dx}(e)=0\right]$$

$$=2x-\frac{8}{x^3}-\frac{2}{3}\sec^2 x.$$

EXAMPLE 2 *Find the derivative of* $\left\{\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7\right\}$.

SOLUTION We have $\dfrac{d}{dx}\left\{\dfrac{3}{\sqrt[3]{x}}-\dfrac{5}{\cos x}+\dfrac{6}{\sin x}-\dfrac{2\tan x}{\sec x}+7\right\}$

$$=\frac{d}{dx}\{3x^{-1/3}-5\sec x+6\operatorname{cosec} x-2\sin x+7\}$$

$$=3\cdot\frac{d}{dx}(x^{-1/3})-5\cdot\frac{d}{dx}(\sec x)+6\cdot\frac{d}{dx}(\operatorname{cosec} x)$$

$$-2\cdot\frac{d}{dx}(\sin x)+\frac{d}{dx}(7)$$

$$=3\cdot\left(-\frac{1}{3}\right)x^{-4/3}-5\sec x\tan x-6\operatorname{cosec} x\cot x-2\cos x$$

$$=\left\{\frac{-1}{x^{4/3}}-5\sec x\tan x-6\operatorname{cosec} x\cot x-2\cos x\right\}$$

EXAMPLE 3 *Differentiate the following functions:*

(i) $(x^2-5x+6)(x-3)$ *(ii)* $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ *(iii)* $\dfrac{3x^2+2x+5}{\sqrt{x}}$

SOLUTION (i) $\dfrac{d}{dx}\{(x^2-5x+6)\,(x-3)\}$

$$=\frac{d}{dx}(x^3-8x^2+21x-18)$$

$$=\frac{d}{dx}(x^3)-8\cdot\frac{d}{dx}(x^2)+21\cdot\frac{d}{dx}(x)-\frac{d}{dx}(18)$$

$$=3x^2-8\times 2x+21\times 1-0=(3x^2-16x+21).$$

(ii) $\dfrac{d}{dx}\left\{\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2\right\}=\dfrac{d}{dx}\left\{\left(x+\dfrac{1}{x}+2\right)\right\}$

$$=\frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})+\frac{d}{dx}(2)$$

$$=1+(-1)x^{-2}+0=\left(1-\frac{1}{x^2}\right).$$

(iii) $\dfrac{d}{dx}\left\{\dfrac{3x^2+2x+5}{\sqrt{x}}\right\}=\dfrac{d}{dx}\{3x^{3/2}+2x^{1/2}+5x^{-1/2}\}$

[on dividing each term by $\sqrt{x}$]

$$=3\cdot\frac{d}{dx}(x^{3/2})+2\cdot\frac{d}{dx}(x^{1/2})+5\cdot\frac{d}{dx}(x^{-1/2})$$

$$=3\cdot\frac{3}{2}x^{1/2}+2\cdot\frac{1}{2}x^{-1/2}+5\cdot\left(-\frac{1}{2}\right)x^{-3/2}$$

$$=\frac{9}{2}\sqrt{x}+\frac{1}{\sqrt{x}}-\frac{5}{2}x^{-3/2}.$$

EXAMPLE 4 *If* $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION $y=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}=\sqrt{\dfrac{2\sin^2 x}{2\cos^2 x}}=\tan x.$

$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}(\tan x)=\sec^2 x.$$

EXAMPLE 5 *If* $y=\left(1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots\infty\right)$, *show that* $\dfrac{dy}{dx}=y.$

SOLUTION We have, $y=e^x.$

$$\therefore\quad \frac{dy}{dx}=\frac{d}{dx}(e^x)=e^x=y.$$

EXAMPLE 6 *If* $u=3t^4-5t^3+2t^2-18t+4$, *find* $\dfrac{du}{dt}$ *at* $t=1.$

SOLUTION $\dfrac{du}{dt}=\dfrac{d}{dt}(3t^4-5t^3+2t^2-18t+4)$

$$=3\cdot\frac{d}{dt}(t^4)-5\cdot\frac{d}{dt}(t^3)+2\cdot\frac{d}{dt}(t^2)-18\cdot\frac{d}{dt}(t)+\frac{d}{dt}(4)$$

$$=3\times 4t^3-5\times 3t^2+2\times 2t-18\times 1+0$$

$$=12t^3-15t^2+4t-18.$$

$$\therefore\quad \left(\frac{du}{dt}\right)_{t=1}=(12\times 1^3-15\times 1^2+4\times 1-18)$$

$$=(12-15+4-18)=-17.$$

EXERCISE 28A

Differentiate the following functions:

1. (i) x^{-3} (ii) $\sqrt[3]{x}$ **2.** (i) $\frac{1}{x}$ (ii) $\frac{1}{\sqrt{x}}$ (iii) $\frac{1}{\sqrt[3]{x}}$

3. (i) $3x^{-5}$ (ii) $\frac{1}{5x}$ (iii) $6\cdot\sqrt[3]{x^2}$

4. (i) $6x^5 + 4x^3 - 3x^2 + 2x - 7$ (ii) $5x^{-3/2} + \frac{4}{\sqrt{x}} + \sqrt{x} - \frac{7}{x}$

(iii) $ax^3 + bx^2 + cx + d$, where a, b, c, d are constants

5. (i) $4x^3 + 3\cdot 2^x + 6\cdot\sqrt[8]{x^{-4}} + 5\cot x$

(ii) $\frac{x}{3} - \frac{3}{x} + \sqrt{x} - \frac{1}{\sqrt{x}} + x^2 - 2^x + 6x^{-2/3} - \frac{2}{3}x^6$

6. (i) $4\cot x - \frac{1}{2}\cos x + \frac{2}{\cos x} - \frac{3}{\sin x} + \frac{6\cot x}{\operatorname{cosec} x} + 9$

(ii) $-5\tan x + 4\tan x\cos x - 3\cot x\sec x + 2\sec x - 13$

7. (i) $(2x + 3)(3x - 5)$ (ii) $x(1 + x)^3$ (iii) $\left(\sqrt{x} + \frac{1}{x}\right)\left(x - \frac{1}{\sqrt{x}}\right)$

(iv) $\left(x - \frac{1}{x}\right)^2$ (v) $\left(x^2 + \frac{1}{x^2}\right)^3$ (vi) $(2x^2 + 5x - 1)(x - 3)$

8. (i) $\frac{3x^2 + 4x - 5}{x}$ (ii) $\frac{(x^3 + 1)(x - 2)}{x^2}$ (iii) $\frac{x - 4}{2\sqrt{x}}$

(iv) $\frac{(1 + x)\sqrt{x}}{\sqrt[3]{x}}$ (v) $\frac{ax^2 + bx + c}{\sqrt{x}}$ (vi) $\frac{a + b\cos x}{\sin x}$

9. (i) If $y = 6x^5 - 4x^4 - 2x^2 + 5x - 9$, find $\frac{dy}{dx}$ at $x = -1$.

(ii) If $y = (\sin x + \tan x)$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{3}$.

(iii) If $y = \frac{(2 - 3\cos x)}{\sin x}$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$.

10. If $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$, show that $2x\cdot\frac{dy}{dx} + y = 2\sqrt{x}$.

11. If $y = \left(\sqrt{\frac{x}{a}} + \sqrt{\frac{a}{x}}\right)$, prove that $(2xy)\left(\frac{dy}{dx}\right) = \left(\frac{x}{a} - \frac{a}{x}\right)$.

12. If $y = \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}}$, find $\frac{dy}{dx}$.

13. $y = \dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$, find $\dfrac{dy}{dx}$.

Hint $\dfrac{1-\tan^2(x/2)}{1+\tan^2(x/2)} = \cos x.$

ANSWERS (EXERCISE 28A)

1. (i) $\dfrac{-3}{x^4}$ (ii) $\dfrac{1}{3}x^{-2/3}$ **2.** (i) $\dfrac{-1}{x^2}$ (ii) $-\dfrac{1}{2}x^{-3/2}$ (iii) $-\dfrac{1}{3}x^{-4/3}$

3. (i) $\dfrac{-15}{x^6}$ (ii) $\dfrac{-1}{5x^2}$ (iii) $4x^{-1/3}$

4. (i) $30x^4 + 12x^2 - 6x + 2$ (ii) $\dfrac{-15}{2}x^{-5/2} - 2x^{-3/2} + \dfrac{1}{2\sqrt{x}} + \dfrac{7}{x^2}$

(iii) $3ax^2 + 2bx + c$

5. (i) $12x^2 + (3\log 2)\cdot 2^x - 3x^{-3/2} - 5\operatorname{cosec}^2 x$

(ii) $\dfrac{1}{3} + \dfrac{3}{x^2} + \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2}x^{-3/2} + 2x - 2^x(\log 2) - 4x^{-5/3} - 4x^5$

6. (i) $-4\operatorname{cosec}^2 x + \dfrac{1}{2}\sin x + 2\sec x \tan x + 3\operatorname{cosec} x \cot x - 6\sin x$

(ii) $-5\sec^2 x + 4\cos x + 3\operatorname{cosec} x \cot x + 2\sec x \tan x$

7. (i) $12x - 1$ (ii) $4x^3 + 9x^2 + 6x + 1$ (iii) $\dfrac{3}{2}(x^{1/2} + x^{-5/2})$

(iv) $2\left(x - \dfrac{1}{x^3}\right)$ (v) $6x^5 - \dfrac{6}{x^7} + 6x - \dfrac{6}{x^3}$ (vi) $6x^2 - 2x - 16$

8. (i) $\left(3 + \dfrac{5}{x^2}\right)$ (ii) $3x^2 - 2 - \dfrac{1}{x^2} + \dfrac{4}{x^3}$

(iii) $\dfrac{1}{4\sqrt{x}} + \dfrac{1}{x\sqrt{x}}$ (iv) $\dfrac{1}{6}x^{-5/6} + \dfrac{7}{6}x^{1/6}$

(v) $\dfrac{3}{2}a\sqrt{x} + \dfrac{b}{2\sqrt{x}} - \dfrac{c}{2}x^{-3/2}$ (vi) $-\operatorname{cosec} x(a\cot x + b\operatorname{cosec} x)$

9. (i) 55 (ii) 9/2 (iii) $2(3-\sqrt{2})$ **12.** $-\operatorname{cosec}^2 x$ **13.** $-\sin x$

Some Derivatives from the First Principle

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate x^6 from the first principle.*

SOLUTION Let $y = x^6$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^6$

$$\Rightarrow \quad \delta y = (x + \delta x)^6 - x^6$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^6 - x^6}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^6 - x^6}{\delta x}$$

$$= \lim_{(x + \delta x) \to 0} \frac{(x + \delta x)^6 - x^6}{(x + \delta x) - x} \quad [\because \ (\delta x \to 0) \ \Rightarrow \ (x + \delta x) \to x]$$

$$= 6x^{(6-1)} = 6x^5 \qquad \left[\because \ \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right].$$

Hence, $\frac{d}{dx}(x^6) = 6x^5$.

EXAMPLE 2 *Differentiate e^{3x} from the first principle.*

SOLUTION Let $y = e^{3x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{3(x + \delta x)}$

$$\Rightarrow \quad \delta y = e^{3(x+\delta x)} - e^{3x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{3(x+\delta x)} - e^{3x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{3(x + \delta x)} - e^{3x}}{\delta x}$$

$$= \lim_{\delta x \to 0} 3e^{3x} \cdot \left(\frac{e^{3\delta x} - 1}{3\delta x}\right)$$

$$= 3e^{3x} \cdot \lim_{z \to 0} \left(\frac{e^z - 1}{z} \right), \text{ where } z = 3\delta x$$

$$= (3e^{3x} \times 1) = 3e^{3x} \qquad \left[\because \lim_{z \to 0} \left(\frac{e^z - 1}{z} \right) = 1 \right].$$

Hence, $\frac{d}{dx}(e^{3x}) = 3e^{3x}$.

EXAMPLE 3 *Find the derivative of each of the following from the first principle.*

(i) $x^2 + 3x + 5$ *(ii)* $x^3 + 4x^2 + 3x + 2$

SOLUTION (i) Let $y = x^2 + 3x + 5$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^2 + 3(x + \delta x) + 5$

$$\Rightarrow \delta y = [(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{[(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[(x + \delta x)^2 + 3(x + \delta x) + 5] - (x^2 + 3x + 5)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\delta x)^2 + 2x \cdot \delta x + 3\delta x}{\delta x} = \lim_{\delta x \to 0} [(\delta x) + 2x + 3] = (2x + 3).$$

$$\therefore \quad \frac{d}{dx}(x^2 + 3x + 5) = (2x + 3).$$

(ii) Let $y = x^3 + 4x^2 + 3x + 2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2$

$$\Rightarrow \delta y = [(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{[(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[(x + \delta x)^3 + 4(x + \delta x)^2 + 3(x + \delta x) + 2] - (x^3 + 4x^2 + 3x + 2)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\delta x)^3 + 3x \cdot \delta x\,(x + \delta x) + 4(\delta x)^2 + 8x \cdot \delta x + 3\delta x}{\delta x}$$

$$= \lim_{\delta x \to 0} [(\delta x)^2 + 3x^2 + 3x \cdot \delta x + 4(\delta x) + 8x + 3] = (3x^2 + 8x + 3).$$

$$\therefore \quad \frac{d}{dx}(x^3 + 4x^2 + 3x + 2) = 3x^2 + 8x + 3.$$

EXAMPLE 4 *Differentiate each of the following ab initio.*

(i) $(x + 5)^7$ *(ii)* $\left(x - \frac{1}{x}\right)$

SOLUTION (i) Let $y = (x + 5)^7$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x + 5)^7$

$$\Rightarrow \delta y = (x + \delta x + 5)^7 - (x + 5)^7$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{(x + \delta x + 5)^7 - (x + 5)^7}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x + 5)^7 - (x + 5)^7}{\delta x}$$

$$= \lim_{(x + \delta x + 5) \to (x + 5)} \frac{[(x + \delta x + 5)^7 - (x + 5)^7]}{[(x + \delta x + 5) - (x + 5)]}$$

$$= \lim_{u \to a} \left(\frac{u^7 - a^7}{u - a}\right), \text{ where } (x + \delta x + 5) = u \text{ and } (x + 5) = a$$

$$= 7a^{(7-1)} = 7(x + 5)^6 \qquad \left[\because \lim_{u \to a} \left(\frac{u^n - a^n}{u - a}\right) = na^{n-1}\right].$$

Hence, $\frac{d}{dx}(x + 5)^7 = 7(x + 5)^6$.

(ii) Let $y = \left(x - \frac{1}{x}\right)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \left[(x + \delta x) - \frac{1}{(x + \delta x)}\right]$

$$\Rightarrow \delta y = \left[(x + \delta x) - \frac{1}{(x + \delta x)}\right] - \left(x - \frac{1}{x}\right)$$

$$\Rightarrow \delta y = \delta x + \left\{\frac{1}{x} - \frac{1}{(x+\delta x)}\right\}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\delta x + \left\{\dfrac{(x+\delta x)-x}{x(x+\delta x)}\right\}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\delta x + \dfrac{\delta x}{x(x+\delta x)}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[1 + \frac{1}{x(x+\delta x)}\right] = \left(1 + \frac{1}{x^2}\right).$$

Hence, $\dfrac{d}{dx}\left(x - \dfrac{1}{x}\right) = \left(1 + \dfrac{1}{x^2}\right)$.

EXAMPLE 5 *Find the derivative of each of the following from the first principle.*

(i) $\sin 2x$ *(ii)* $\cos 3x$ *(iii)* $\tan 5x$

SOLUTION (i) Let $y = \sin 2x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin 2(x + \delta x)$

$$\Rightarrow \delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sin(2x + 2\delta x) - \sin 2x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\sin(2x + 2\delta x) - \sin 2x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos(2x + \delta x)\sin \delta x}{\delta x}$$

$$\left\{\because (\sin C - \sin D) = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right\}$$

$$= 2 \cdot \lim_{\delta x \to 0} \cos(2x + \delta x) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= (2\cos 2x \times 1) = 2\cos 2x.$$

Hence, $\dfrac{d}{dx}(\sin 2x) = 2\cos 2x$.

(ii) Let $y = \cos 3x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos 3(x + \delta x)$

$\Rightarrow \delta y = \cos(3x + 3\delta x) - \cos 3x$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\cos(3x + 3\delta x) - \cos 3x}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{[\cos(3x + 3\delta x) - \cos 3x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(3x + \frac{3}{2}\delta x\right)\sin\left(\frac{3}{2}\delta x\right)}{\delta x}$$

$$\left\{\because \ (\cos C - \cos D) = -2\sin\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)\right\}$$

$$= -\lim_{\delta x \to 0} \sin\left(3x + \frac{3}{2}\delta x\right) \cdot \lim_{\delta x \to 0} \frac{3\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)}$$

$$= (-\sin 3x \times 3 \times 1) = -3\sin 3x.$$

Hence, $\dfrac{d}{dx}(\cos 3x) = -3\sin 3x.$

(iii) Let $y = \tan 5x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan 5(x + \delta x)$

$\Rightarrow \delta y = \tan(5x + 5\delta x) - \tan 5x$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\tan(5x + 5\delta x) - \tan 5x}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\tan(5x + 5\delta x) - \tan 5x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin(5x + 5\delta x)}{\cos(5x + 5\delta x)} - \frac{\sin 5x}{\cos 5x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(5x + 5\delta x)\cos 5x - \cos(5x + 5\delta x)\sin 5x}{\cos(5x + 5\delta x) \cdot \cos 5x \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(5x + 5\delta x - 5x)}{\cos(5x + 5\delta x) \cdot \cos 5x \cdot \delta x}$$

$$[\because \ \sin A \cos B - \cos A \sin B = \sin(A - B)]$$

$$= \frac{1}{\cos 5x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos(5x + 5\delta x)} \cdot \lim_{\delta x \to 0} \frac{\sin(5\delta x) \times 5}{(5\delta x)}$$

$$= \left[\frac{1}{\cos 5x} \times \frac{1}{\cos 5x} \times (1 \times 5)\right] = \frac{5}{\cos^2 5x} = 5\sec^2 5x.$$

Hence, $\frac{d}{dx}(\tan 5x) = 5\sec^2 5x.$

EXAMPLE 6 *Differentiate* $x \sin x$ *from the first principle.*

SOLUTION Let $y = x \sin x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)\sin(x + \delta x)$

$$\Rightarrow \quad \delta y = (x + \delta x)\sin(x + \delta x) - x \sin x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)\sin(x + \delta x) - x \sin x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{(x + \delta x)\sin(x + \delta x) - x \sin x\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{x\{\sin(x + \delta x) - \sin x\}}{\delta x} + \sin(x + \delta x)\right]$$

$$= \lim_{\delta x \to 0} \frac{x\{\sin(x + \delta x) - \sin x\}}{\delta x} + \lim_{\delta x \to 0} \sin(x + \delta x)$$

$$= \lim_{\delta x \to 0} \left\{\frac{2x \cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}\right\} + \sin x$$

$$= \left\{x \cdot \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}\right\} + \sin x$$

$$= (x \cdot \cos x \cdot 1) + \sin x = (x \cos x + \sin x).$$

Hence, $\frac{d}{dx}(x \sin x) = (x \cos x + \sin x).$

EXAMPLE 7 *Differentiate* $\frac{\sin x}{x}$ *from the first principle.* **[CBSE 2002]**

SOLUTION Let $y = \frac{\sin x}{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \frac{\sin (x + \delta x)}{(x + \delta x)}$

$$\Rightarrow \quad \delta y = \left\{\frac{\sin (x + \delta x)}{(x + \delta x)} - \frac{\sin x}{x}\right\} = \frac{\{x \sin (x + \delta x) - (x + \delta x) \sin x\}}{x(x + \delta x)}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\{x \sin (x + \delta x) - (x + \delta x) \sin x\}}{x(x + \delta x) \cdot \delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{x \sin (x + \delta x) - (x + \delta x) \sin x\}}{x (x + \delta x) \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{x[\sin (x + \delta x) - \sin x] - (\delta x) \sin x}{x (x + \delta x) \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \frac{x\left[2\cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)\right] - (\delta x) \sin x}{x (x + \delta x) \cdot \delta x}$$

$$= \left\{\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin (\delta x/2)}{(\delta x/2)} \cdot \lim_{\delta x \to 0} \frac{1}{(x + \delta x)}\right\}$$

$$- \lim_{\delta x \to 0} \frac{\sin x}{x(x + \delta x)}$$

$$= \left(\cos x \times 1 \times \frac{1}{x}\right) - \frac{\sin x}{x^2} = \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) = \frac{(x \cos x - \sin x)}{x^2}.$$

Hence, $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{(x \cos x - \sin x)}{x^2}$.

EXAMPLE 8 *Differentiate* $\cot (2x + 1)$ *from the first principle.*

SOLUTION Let $y = \cot (2x + 1)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cot [2(x + \delta x) + 1]$

$$\Rightarrow \quad \delta y = \cot [2(x + \delta x) + 1] - \cot (2x + 1)$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\cot [2(x + \delta x) + 1] - \cot (2x + 1)}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cot(2x + 2\delta x + 1) - \cot(2x + 1)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\cos(2x + 2\delta x + 1)}{\sin(2x + 2\delta x + 1)} - \frac{\cos(2x + 1)}{\sin(2x + 1)}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x + 1)\cos(2x + 2\delta x + 1) - \cos(2x + 1)\sin(2x + 2\delta x + 1)}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$$= \lim_{\delta x \to 0} \frac{\sin[(2x + 1) - (2x + 2\delta x + 1)]}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$

$$= \lim_{\delta x \to 0} \frac{\sin(-2\delta x)}{\delta x \cdot \sin(2x + 2\delta x + 1) \cdot \sin(2x + 1)}$$

$$= \frac{1}{\sin(2x+1)} \cdot \lim_{\delta x \to 0} \left\{\frac{-\sin(2\delta x)}{2\delta x} \times 2\right\} \cdot \lim_{\delta x \to 0} \frac{1}{\sin(2x + 2\delta x + 1)}$$

$$= \frac{1}{\sin(2x + 1)} \cdot (-1) \times 2 \times \frac{1}{\sin(2x + 1)} = \frac{-2}{\sin^2(2x + 1)}$$

$$= -2\operatorname{cosec}^2(2x + 1).$$

Hence, $\frac{d}{dx}[\cot(2x + 1)] = -2\operatorname{cosec}^2(2x + 1).$

EXAMPLE 9 *Find the derivative of each of the following from the first principle.*

(i) $\sqrt{2x + 3}$ *(ii)* $\sqrt{4 - x}$ *(iii)* $\frac{1}{\sqrt{x}}$

SOLUTION (i) Let $y = \sqrt{2x + 3}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{2(x + \delta x) + 3}$

$$\Rightarrow \delta y = \sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\sqrt{2(x + \delta x) + 3} - \sqrt{2x + 3}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{(\sqrt{2x + 2\delta x + 3} - \sqrt{2x + 3})}{\delta x} \times \frac{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}\right\}$$

$$= \lim_{\delta x \to 0} \frac{\{(2x + 2\delta x + 3) - (2x + 3)\}}{\delta x \cdot (\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \lim_{\delta x \to 0} \frac{2\delta x}{\delta x \cdot (\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \lim_{\delta x \to 0} \frac{2}{(\sqrt{2x + 2\delta x + 3} + \sqrt{2x + 3})}$$

$$= \frac{2}{2\sqrt{2x + 3}} = \frac{1}{\sqrt{2x + 3}}.$$

$$\therefore \quad \frac{d}{dx}(\sqrt{2x + 3}) = \frac{1}{\sqrt{2x + 3}}.$$

(ii) Let $y = \sqrt{4 - x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{4 - (x + \delta x)}$

$$\Rightarrow \delta y = \sqrt{4 - (x + \delta x)} - \sqrt{4 - x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{4 - (x + \delta x)} - \sqrt{4 - x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{4 - (x + \delta x)} - \sqrt{4 - x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{(\sqrt{4 - x - \delta x} - \sqrt{4 - x})}{\delta x} \times \frac{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})}{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})} \right\}$$

$$= \lim_{\delta x \to 0} \frac{\{(4 - x - \delta x) - (4 - x)\}}{\delta x\,(\sqrt{4 - x - \delta x} + \sqrt{4 - x})}$$

$$= \lim_{\delta x \to 0} \frac{-\delta x}{\delta x\,(\sqrt{4 - x - \delta x} + \sqrt{4 - x})}$$

$$= \lim_{\delta x \to 0} \frac{-1}{(\sqrt{4 - x - \delta x} + \sqrt{4 - x})} = \frac{-1}{2\sqrt{4 - x}}.$$

$$\therefore \quad \frac{d}{dx}(\sqrt{4 - x}) = \frac{-1}{2\sqrt{4 - x}}.$$

(iii) Let $y = \frac{1}{\sqrt{x}}$.

Let δy be an increment in y, corresponding to an increment δx in x.

$$\text{Then, } y + \delta y = \frac{1}{\sqrt{x + \delta x}}$$

$$\Rightarrow \delta y = \left(\frac{1}{\sqrt{x + \delta x}} - \frac{1}{\sqrt{x}} \right)$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{1}{\delta x} \cdot \left[\frac{1}{\sqrt{x + \delta x}} - \frac{1}{\sqrt{x}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \cdot \left[\frac{1}{\sqrt{x + \delta x}} - \frac{1}{\sqrt{x}} \right]$$

$$= \lim_{\delta x \to 0} \left\{ \frac{(\sqrt{x} - \sqrt{x + \delta x})}{\delta x \cdot \sqrt{x + \delta x} \cdot \sqrt{x}} \times \frac{(\sqrt{x} + \sqrt{x + \delta x})}{(\sqrt{x} + \sqrt{x + \delta x})} \right\}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{\{x - (x + \delta x)\}}{\delta x \cdot \sqrt{x + \delta x} \cdot \sqrt{x} \cdot (\sqrt{x} + \sqrt{x + \delta x})} \right\}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{-\delta x}{\delta x \cdot \sqrt{x + \delta x} \cdot \sqrt{x} \cdot (\sqrt{x} + \sqrt{x + \delta x})} \right\}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{-1}{\sqrt{x + \delta x} \cdot \sqrt{x} \cdot (\sqrt{x} + \sqrt{x + \delta x})} \right\}$$

$$= \frac{-1}{\sqrt{x} \cdot \sqrt{x} \cdot (2\sqrt{x})} = \frac{-1}{2x^{3/2}}.$$

$$\text{Hence, } \frac{d}{dx}\left(\frac{1}{\sqrt{x}} \right) = \frac{-1}{2x^{3/2}}.$$

EXAMPLE 10 *Find the derivative of $x^{-3/2}$ from the first principle.*

SOLUTION Let $y = x^{-3/2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^{-3/2}$

$$\Rightarrow \quad \delta y = (x + \delta x)^{-3/2} - x^{-3/2}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\{(x + \delta x)^{-3/2} - x^{-3/2}\}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-3/2} - x^{-3/2}\}}{\delta x} = \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-3/2} - x^{-3/2}\}}{\{(x + \delta x) - x\}}$$

$$= \lim_{u \to x} \frac{(u^{-3/2} - x^{-3/2})}{(u - x)}, \text{ where } u = (x + \delta x)$$

$$= -\frac{3}{2} x^{\left(-\frac{3}{2}-1\right)} \qquad \left[\because \lim_{u \to a} \frac{u^n - a^n}{u - a} = na^{n-1}\right]$$

$$= -\frac{3}{2} x^{-5/2}.$$

Hence, $\frac{d}{dx}(x^{-3/2}) = -\frac{3}{2} x^{-5/2}$.

EXAMPLE 11 *Find the derivative of* $\frac{1}{x^2}$ *from the first principle.*

SOLUTION Let $y = \frac{1}{x^2}$. Then, $y = x^{-2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^{-2}$

$\Rightarrow \quad \delta y = (x + \delta x)^{-2} - x^{-2}$

$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{\delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{\delta x} = \lim_{\delta x \to 0} \frac{\{(x + \delta x)^{-2} - x^{-2}\}}{[(x + \delta x) - x]}$$

$$= \lim_{u \to x} \frac{(u^{-2} - x^{-2})}{(u - x)}, \text{ where } u = (x + \delta x)$$

$$= -2x^{(-2-1)} = -2x^{-3} = \frac{-2}{x^3} \qquad \left[\because \lim_{u \to a} \frac{u^n - a^n}{u - a} = na^{n-1}\right].$$

Hence, $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$.

EXAMPLE 12 *Differentiate* $\sqrt{\sin 3x}$ *from the first principle.* **[CBSE 2004C]**

SOLUTION Let $y = \sqrt{\sin 3x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\sin 3(x + \delta x)}$

$\Rightarrow \quad \delta y = \sqrt{\sin (3x + 3\delta x)} - \sqrt{\sin 3x}$

$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\sqrt{\sin (3x + 3\delta x)} - \sqrt{\sin 3x}}{\delta x}$

$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \left[\frac{\{\sqrt{\sin(3x+3\delta x)} - \sqrt{\sin 3x}\}}{\delta x} \times \frac{\{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}}{\{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}} \right]$$

$$= \lim_{\delta x \to 0} \frac{[\sin(3x+3\delta x) - \sin 3x]}{\delta x \cdot \{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(3x + \frac{3}{2}\delta x\right) \sin\left(\frac{3}{2}\delta x\right)}{\delta x \cdot \{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}}$$

$$\left[\because (\sin C - \sin D) = 2\cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \left[2\cos\left(3x + \frac{3}{2}\delta x\right) \cdot \frac{\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)} \times \frac{3}{2} \times \frac{1}{\{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}} \right]$$

$$= 3 \cdot \lim_{\delta x \to 0} \cos\left(3x + \frac{3}{2}\delta x\right) \times \lim_{\frac{3}{2}\delta x \to 0} \frac{\sin\left(\frac{3}{2}\delta x\right)}{\left(\frac{3}{2}\delta x\right)} \times \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin(3x+3\delta x)} + \sqrt{\sin 3x}\}}$$

$$= 3\cos 3x \times 1 \times \frac{1}{2\sqrt{\sin 3x}} = \frac{3\cos 3x}{2\sqrt{\sin 3x}}.$$

Hence, $\frac{d}{dx}(\sqrt{\sin 3x}) = \frac{3\cos 3x}{2\sqrt{\sin 3x}}$.

EXAMPLE 13 *Find the derivative of $e^{\sqrt{x}}$ from the first principle.* **[CBSE 2003C]**

SOLUTION Let $y = e^{\sqrt{x}}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{\sqrt{x+\delta x}}$

$$\Rightarrow \quad \delta y = e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}}{\delta x} = \lim_{\delta x \to 0} \frac{e^{\sqrt{x}}\,[e^{\sqrt{x+\delta x} - \sqrt{x}} - 1]}{\{(x+\delta x) - x\}}$$

$$= \lim_{\delta x \to 0} \frac{e^{\sqrt{x}}\,[e^{\sqrt{x+\delta x} - \sqrt{x}} - 1]}{(\sqrt{x+\delta x} - \sqrt{x})(\sqrt{x+\delta x} + \sqrt{x})}$$

$$= e^{\sqrt{x}} \cdot \lim_{\theta \to 0} \frac{(e^{\theta} - 1)}{\theta} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})},$$

where $\theta = (\sqrt{x + \delta x} - \sqrt{x})$ [clearly, $\theta \to 0$ when $\delta x \to 0$]

$$= \left(e^{\sqrt{x}} \times 1 \times \frac{1}{2\sqrt{x}} \right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \quad \left[\because \lim_{\theta \to 0} \frac{(e^{\theta} - 1)}{\theta} = 1 \right].$$

Hence, $\frac{d}{dx}(e^{\sqrt{x}}) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$.

EXAMPLE 14 *Find the derivative of e^{x^2} from the first principle.* **[CBSE 2002C]**

SOLUTION Let $y = e^{x^2}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{(x + \delta x)^2}$

$$\Rightarrow \quad \delta y = e^{(x + \delta x)^2} - e^{x^2}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{(x + \delta x)^2} - e^{x^2}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{(x + \delta x)^2} - e^{x^2}}{\delta x} = \lim_{\delta x \to 0} \frac{e^{(x^2 + 2x\delta x + \delta x^2)} - e^{x^2}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^{x^2}[e^{(2x\delta x + \delta x^2)} - 1]}{\delta x}$$

$$= e^{x^2} \cdot \lim_{\delta x \to 0} \frac{(e^{2x\delta x + \delta x^2} - 1)}{(2x\delta x + \delta x^2)} \times \frac{(2x\delta x + \delta x^2)}{\delta x}$$

$$= e^{x^2} \cdot \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta} \right) \cdot \lim_{\delta x \to 0} (2x + \delta x), \text{ where } (2x\delta x + \delta x^2) = \theta$$

[clearly, $\theta \to 0$ when $\delta x \to 0$]

$$= (e^{x^2} \times 1 \times 2x) = 2xe^{x^2} \qquad \left[\because \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta} \right) = 1 \right].$$

Hence, $\frac{d}{dx}(e^{x^2}) = 2xe^{x^2}$.

EXAMPLE 15 *Find the derivative of $e^{\sin x}$ from the first principle.*

SOLUTION Let $y = e^{\sin x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = e^{\sin (x + \delta x)}$

$$\Rightarrow \quad \delta y = e^{\sin\,(x+\delta x)} - e^{\sin\,x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{\sin\,(x+\delta x)} - e^{\sin\,x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{[e^{\sin\,(x+\delta x)} - e^{\sin\,x}]}{\delta x} = \lim_{\delta x \to 0} \frac{e^{\sin\,x} \cdot [e^{\sin\,(x+\delta x)} - \sin\,x - 1]}{\delta x}$$

$$= e^{\sin\,x} \cdot \lim_{\delta x \to 0} \frac{[e^{\sin\,(x+\delta x) - \sin\,x} - 1]}{\{\sin\,(x+\delta x) - \sin\,x\}} \cdot \frac{[\sin\,(x+\delta x) - \sin\,x]}{\delta x}$$

$$= e^{\sin\,x} \cdot \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\,(x+\delta x) - \sin\,x}{\delta x},$$

where $\theta = \sin\,(x+\delta x) - \sin\,x \quad [\text{clearly}, (\delta x \to 0) \Rightarrow (\theta \to 0)]$

$$= e^{\sin\,x} \cdot 1 \cdot \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= e^{\sin\,x} \cdot \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$= (e^{\sin\,x} \cdot \cos\,x \cdot 1) = e^{\sin\,x} \cos\,x.$$

Hence, $\frac{d}{dx}(e^{\sin\,x}) = e^{\sin\,x} \cos\,x.$

EXAMPLE 16 *Differentiate each of the following from the first principle.*

(i) $\sqrt{\sin\,x}$ **[CBSE 2004C]** *(ii)* $\sqrt{\cos\,x}$ **[CBSE 2004]**

(iii) $\sqrt{\tan\,x}$ **[CBSE 2002, '04C]** *(iv)* $\sqrt{\text{cosec}\,x}$ **[CBSE 1998]**

SOLUTION (i) Let $y = \sqrt{\sin\,x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\sin\,(x+\delta x)}$

$$\Rightarrow \quad \delta y = \sqrt{\sin\,(x+\delta x)} - \sqrt{\sin\,x}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\sqrt{\sin\,(x+\delta x)} - \sqrt{\sin\,x}}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\sin\,(x+\delta x)} - \sqrt{\sin\,x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\sin\,(x+\delta x)} - \sqrt{\sin\,x}\}}{\delta x} \times \frac{\{\sqrt{\sin\,(x+\delta x)} + \sqrt{\sin\,x}\}}{\{\sqrt{\sin\,(x+\delta x)} + \sqrt{\sin\,x}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\sin(x+\delta x) - \sin x\}}{\delta x \cdot \{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\}}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\} \cdot \delta x}$$

$$= \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin(x+\delta x)} + \sqrt{\sin x}\}}$$

$$= \left\{\cos x \cdot 1 \cdot \frac{1}{2\sqrt{\sin x}}\right\} = \frac{\cos x}{2\sqrt{\sin x}}.$$

Hence, $\frac{d}{dx}(\sqrt{\sin x}) = \frac{\cos x}{2\sqrt{\sin x}}$.

(ii) Let $y = \sqrt{\cos x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\cos(x+\delta x)}$

$$\Rightarrow \delta y = \sqrt{\cos(x+\delta x)} - \sqrt{\cos x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}\}}{\delta x} \times \frac{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\cos(x+\delta x) - \cos x\}}{\delta x \cdot \{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x \cdot \{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)\right]$$

$$= -\lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}\}}$$

$$= (-\sin x) \times 1 \times \frac{1}{2\sqrt{\cos x}} = \frac{-\sin x}{2\sqrt{\cos x}}.$$

Hence, $\frac{d}{dx}(\sqrt{\cos x}) = \frac{-\sin x}{2\sqrt{\cos x}}$.

(iii) Let $y = \sqrt{\tan x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\tan (x + \delta x)}$

$\Rightarrow \delta y = \sqrt{\tan (x + \delta x)} - \sqrt{\tan x}$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{\tan (x + \delta x)} - \sqrt{\tan x}}{\delta x}$

$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$= \lim_{\delta x \to 0} \frac{\sqrt{\tan (x + \delta x)} - \sqrt{\tan x}}{\delta x}$

$= \lim_{\delta x \to 0} \left\{ \frac{\sqrt{\tan (x + \delta x)} - \sqrt{\tan x}}{\delta x} \times \frac{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}}{\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}} \right\}$

$= \lim_{\delta x \to 0} \frac{\tan (x + \delta x) - \tan x}{\delta x [\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}]}$

$= \lim_{\delta x \to 0} \frac{\left\{ \frac{\sin (x + \delta x)}{\cos (x + \delta x)} - \frac{\sin x}{\cos x} \right\}}{\delta x [\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}]}$

$= \lim_{\delta x \to 0} \frac{\sin (x + \delta x) \cos x - \cos (x + \delta x) \sin x}{\cos (x + \delta x) \cos x \cdot \delta x \cdot [\sqrt{\tan (x + \delta x)} + \sqrt{\tan x}]}$

$= \lim_{\delta x \to 0} \frac{\sin (x + \delta x - x)}{\cos (x + \delta x) \cdot \cos x \cdot \delta x \cdot (\sqrt{\tan (x + \delta x)} + \sqrt{\tan x})}$

$= \frac{1}{\cos x} \cdot \lim_{\delta x \to 0} \frac{1}{\cos (x + \delta x)} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$

$\cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{\tan (x + \delta x)} + \sqrt{\tan x})}$

$= \left(\frac{1}{\cos x} \cdot \frac{1}{\cos x} \cdot 1 \cdot \frac{1}{2\sqrt{\tan x}} \right) = \frac{\sec^2 x}{2\sqrt{\tan x}}$.

Hence, $\frac{d}{dx}(\sqrt{\tan x}) = \frac{\sec^2 x}{2\sqrt{\tan x}}$.

(iv) Let $y = \sqrt{\operatorname{cosec} x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sqrt{\operatorname{cosec} (x + \delta x)}$

$$\Rightarrow \delta y = \sqrt{\text{cosec}(x+\delta x)} - \sqrt{\text{cosec}\, x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\sqrt{\text{cosec}\,(x+\delta x)} - \sqrt{\text{cosec}\, x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\text{cosec}\,(x+\delta x)} - \sqrt{\text{cosec}\, x}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{1}{\sqrt{\sin(x+\delta x)}} - \frac{1}{\sqrt{\sin x}}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\sqrt{\sin x} - \sqrt{\sin(x+\delta x)}\}}{\delta x \cdot \sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}} \times \frac{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{\{\sin x - \sin(x+\delta x)\}}{\{\sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}\}} \times \frac{1}{\delta x \cdot \{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\{\sqrt{\sin(x+\delta x)}\} \cdot \sqrt{\sin x}\} \cdot \delta x \cdot \{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= -\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot$$

$$\lim_{\delta x \to 0} \frac{1}{\sqrt{\sin(x+\delta x)} \cdot \sqrt{\sin x}} \cdot \lim_{\delta x \to 0} \frac{1}{\{\sqrt{\sin x} + \sqrt{\sin(x+\delta x)}\}}$$

$$= -\cos x \times 1 \times \frac{1}{\sqrt{\sin x} \cdot \sqrt{\sin x}} \cdot \frac{1}{(\sqrt{\sin x} + \sqrt{\sin x})}$$

$$= \frac{-\cos x}{\sin x} \cdot \frac{1}{2\sqrt{\sin x}} = -\frac{1}{2}\sqrt{\text{cosec}\, x}\ \cot x.$$

Hence, $\frac{d}{dx}(\sqrt{\text{cosec}\, x}) = -\frac{1}{2}\sqrt{\text{cosec}\, x}\ \cot x.$

EXAMPLE 17 *Differentiate each of the following from the first principle.*
(i) $\sin\sqrt{x}$ **[CBSE 2004C]** *(ii)* $\cos\sqrt{x}$
(iii) $\tan\sqrt{x}$ **[CBSE 2002, '04]**

SOLUTION (i) Let $y = \sin\sqrt{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin\sqrt{x + \delta x}$

$\Rightarrow \delta y = \sin\sqrt{x + \delta x} - \sin\sqrt{x}$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\sin\sqrt{x + \delta x} - \sin\sqrt{x}}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{\sin\sqrt{x + \delta x} - \sin\sqrt{x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\dfrac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{\{(x + \delta x) - x\}}$$

$$\left[\text{using } \sin C - \sin D = 2\cos\left(\frac{C + D}{2}\right)\sin\left(\frac{C - D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left(\dfrac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right)\sin\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{(\sqrt{x + \delta x} + \sqrt{x})\,(\sqrt{x + \delta x} - \sqrt{x})}$$

$$[\because\ (x + \delta x) - x = (\sqrt{x + \delta x} + \sqrt{x})(\sqrt{x + \delta x} - \sqrt{x})]$$

$$= \lim_{\delta x \to 0} \frac{\sin\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)}{\left(\dfrac{\sqrt{x + \delta x} - \sqrt{x}}{2}\right)} \cdot \lim_{\delta x \to 0} \cos\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\delta x \to 0} \cos\left(\frac{\sqrt{x + \delta x} + \sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x + \delta x} + \sqrt{x})},$$

where $\dfrac{(\sqrt{x + \delta x} - \sqrt{x})}{2} = \theta$ and $\delta x \to 0 \Rightarrow \theta \to 0$

$$= 1 \times \cos\sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2\sqrt{x}}.$$

Hence, $\dfrac{d}{dx}(\sin\sqrt{x}) = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

(ii) Let $y = \cos\sqrt{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos\sqrt{x + \delta x}$

$\Rightarrow \delta y = \cos\sqrt{x + \delta x} - \cos\sqrt{x}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\{\cos\sqrt{x+\delta x} - \cos\sqrt{x}\}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{\cos\sqrt{x+\delta x} - \cos\sqrt{x}\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\{(x+\delta x)-x\}}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{(\sqrt{x+\delta x}+\sqrt{x})(\sqrt{x+\delta x}-\sqrt{x})}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})}$$

$$= -\lim_{\delta x \to 0} \sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right) \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x}+\sqrt{x})},$$

where $\frac{(\sqrt{x+\delta x}-\sqrt{x})}{2} = \theta$

$$= -\sin\left(\frac{2\sqrt{x}}{2}\right) \times 1 \times \frac{1}{2\sqrt{x}} = \frac{-\sin\sqrt{x}}{2\sqrt{x}}.$$

Hence, $\frac{d}{dx}(\cos\sqrt{x}) = \frac{-\sin\sqrt{x}}{2\sqrt{x}}$.

(iii) Let $y = \tan\sqrt{x}$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan\sqrt{x+\delta x}$

$$\Rightarrow \delta y = \tan\sqrt{x+\delta x} - \tan\sqrt{x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\tan\sqrt{x+\delta x} - \tan\sqrt{x}}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\tan\sqrt{x+\delta x} - \tan\sqrt{x})}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin\sqrt{x+\delta x}}{\cos\sqrt{x+\delta x}} - \frac{\sin\sqrt{x}}{\cos\sqrt{x}}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(\sin\sqrt{x+\delta x}\cos\sqrt{x} - \cos\sqrt{x+\delta x}\sin\sqrt{x})}{\delta x \cdot \cos\sqrt{x+\delta x} \cdot \cos\sqrt{x}}$$

$$= \lim_{\delta x \to 0} \left\{\frac{\sin(\sqrt{x+\delta x} - \sqrt{x})}{\{(x+\delta x) - x\}} \cdot \frac{1}{\cos\sqrt{x+\delta x} \cdot \cos\sqrt{x}}\right\}$$

$$[\because \ \delta x = (x+\delta x) - x]$$

$$= \lim_{\delta x \to 0} \left\{\frac{\sin(\sqrt{x+\delta x} - \sqrt{x})}{(\sqrt{x+\delta x} - \sqrt{x})(\sqrt{x+\delta x} + \sqrt{x})} \cdot \frac{1}{\cos\sqrt{x+\delta x} \cdot \cos\sqrt{x}}\right\}$$

$$= \lim_{\delta x \to 0} \frac{\sin(\sqrt{x+\delta x} - \sqrt{x})}{(\sqrt{x+\delta x} - \sqrt{x})} \cdot \lim_{\delta x \to 0} \frac{1}{(\sqrt{x+\delta x} + \sqrt{x})} \cdot \lim_{\delta x \to 0} \frac{1}{\cos\sqrt{x+\delta x} \cdot \cos\sqrt{x}}$$

$$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right) \cdot \frac{1}{2\sqrt{x}} \cdot \frac{1}{\cos^2\sqrt{x}} = 1 \times \frac{1}{2\sqrt{x}} \times \frac{1}{\cos^2\sqrt{x}}$$

$$= \frac{\sec^2\sqrt{x}}{2\sqrt{x}} \cdot$$

Hence, $\frac{d}{dx}(\tan\sqrt{x}) = \frac{\sec^2\sqrt{x}}{2\sqrt{x}}$.

EXAMPLE 18 *Find the derivative of the following from the first principle.*

(i) $\sin^2 x$ *(ii)* $\cos^2 x$ **[CBSE 2003]**

SOLUTION (i) Let $y = \sin^2 x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin^2(x + \delta x)$

$\Rightarrow \ \delta y = \sin^2(x+\delta x) - \sin^2 x$

$\Rightarrow \ \frac{\delta y}{\delta x} = \frac{\sin^2(x+\delta x) - \sin^2 x}{\delta x}$

$\Rightarrow \ \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\sin^2(x+\delta x) - \sin^2 x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x + \delta x)\sin \delta x}{\delta x}$$

$$[\because \sin^2 A - \sin^2 B = \sin(A + B) \cdot \sin(A - B)]$$

$$= \lim_{\delta x \to 0} \sin(2x + \delta x) \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= (\sin 2x \times 1) = \sin 2x.$$

Hence, $\frac{d}{dx}(\sin^2 x) = \sin 2x.$

(ii) Let $y = \cos^2 x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos^2(x + \delta x)$

$$\Rightarrow \delta y = \cos^2(x + \delta x) - \cos^2 x$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\cos^2(x + \delta x) - \cos^2 x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\cos^2(x + \delta x) - \cos^2 x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\cos(x + \delta x) + \cos x] \times [\cos(x + \delta x) - \cos x]}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{2\cos\left(x + \frac{\delta x}{2}\right)\cos\left(\frac{\delta x}{2}\right)\right\}\left\{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)\right\}}{\delta x}$$

$$= -2 \lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \cos\left(\frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin(\delta x/2)}{(\delta x/2)}$$

$$= (-2\cos x \times 1 \times \sin x \times 1) = -2 \sin x \cos x = -\sin 2x.$$

Hence, $\frac{d}{dx}(\cos^2 x) = -\sin 2x.$

EXAMPLE 19 *Find the derivative of the following from the first principle:*

(i) $\sin x^2$ **[CBSE 2001C]** *(ii)* $\cos(x^2 + 1)$ **[CBSE 2001]**

(iii) $\tan x^2$ **[CBSE 2002]**

SOLUTION (i) Let $y = \sin x^2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \sin (x + \delta x)^2$

$\Rightarrow \delta y = \sin (x + \delta x)^2 - \sin x^2$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\sin (x + \delta x)^2 - \sin x^2}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\sin (x + \delta x)^2 - \sin x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{2\cos\left[\dfrac{(x + \delta x)^2 + x^2}{2}\right] \sin\left[\dfrac{(x + \delta x)^2 - x^2}{2}\right]}{\delta x}$$

$$\left[\text{using } (\sin C - \sin D) = 2\cos\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)\right]$$

$$= \lim_{\delta x \to 0} 2\cos\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \frac{\sin\left[\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x\right]}{\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x} \cdot \left(x + \frac{\delta x}{2}\right)$$

$$= 2 \cdot \lim_{\delta x \to 0} \cos\left[\frac{(x + \delta x)^2 + x^2}{2}\right] \cdot \lim_{\delta x \to 0} \frac{\sin\left[\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x\right]}{\left(x + \dfrac{\delta x}{2}\right) \cdot \delta x} \cdot \lim_{\delta x \to 0} \left(x + \frac{\delta x}{2}\right)$$

$= [2 \times \cos x^2 \times 1 \times x] = 2x \cos x^2$.

Hence, $\dfrac{d}{dx} (\sin x^2) = 2x \cos x^2$.

(ii) Let $y = \cos (x^2 + 1)$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \cos [(x + \delta x)^2 + 1]$

$\Rightarrow \delta y = \cos [(x + \delta x)^2 + 1] - \cos (x^2 + 1)$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\cos [(x + \delta x)^2 + 1] - \cos (x^2 + 1)}{\delta x}$

$\Rightarrow \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\{\cos[(x+\delta x)^2+1] - \cos(x^2+1)\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-2\sin\left[\frac{\{(x+\delta x)^2+1+(x^2+1)\}}{2}\right] \cdot \sin\left[\frac{\{(x+\delta x)^2+1-(x^2+1)\}}{2}\right]}{\delta x}$$

$$= -2 \cdot \lim_{\delta x \to 0} \sin\left[x^2 + x\cdot\delta x + 1 + \frac{(\delta x)^2}{2}\right] \cdot \frac{\sin\left[\left(x+\frac{\delta x}{2}\right)\cdot\delta x\right]}{\left[\left(x+\frac{\delta x}{2}\right)\cdot\delta x\right]} \cdot \left(x+\frac{\delta x}{2}\right)$$

$$= -2 \cdot \lim_{\delta x \to 0} \sin\left[x^2 + x\cdot\delta x + 1 + \frac{(\delta x)^2}{2}\right] \cdot \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\delta x \to 0}\left(x+\frac{\delta x}{2}\right),$$

$$\text{where } \theta = \left(x+\frac{\delta x}{2}\right)\cdot\delta x; \text{ clearly, } [\delta x \to 0] \Rightarrow [\theta \to 0]$$

$$= -2\sin(x^2+1) \times 1 \times x$$

$$= -2x\sin(x^2+1).$$

Hence, $\frac{d}{dx}[\cos(x^2+1)] = -2x\sin(x^2+1)$.

(iii) Let $y = \tan x^2$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = \tan(x+\delta x)^2$

$$\Rightarrow \delta y = \tan(x+\delta x)^2 - \tan x^2$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{\tan(x+\delta x)^2 - \tan x^2}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\tan(x+\delta x)^2 - \tan x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin(x+\delta x)^2}{\cos(x+\delta x)^2} - \frac{\sin x^2}{\cos x^2}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{[\sin(x+\delta x)^2 \cos x^2 - \cos(x+\delta x)^2 \sin x^2]}{\delta x \cdot \cos(x+\delta x)^2 \cdot \cos x^2}$$

$$= \lim_{\delta x \to 0} \frac{\sin[(x+\delta x)^2 - x^2]}{\delta x \cdot \cos(x+\delta x)^2 \cdot \cos x^2}$$

$$= \lim_{\delta x \to 0} \frac{\sin[(2x+\delta x)\cdot\delta x]}{[(2x+\delta x)\cdot\delta x]} \cdot \frac{(2x+\delta x)}{\cos(x+\delta x)^2 \cdot \cos x^2}$$

$$= \left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right) \cdot \lim_{\delta x \to 0}(2x+\delta x) \cdot \lim_{\delta x \to 0} \frac{1}{\cos(x+\delta x)^2 \cdot \cos x^2},$$

$$\text{where } (2x+\delta x)\cdot\delta x = \theta \quad [\text{clearly}, (\delta x \to 0) \Rightarrow (\theta \to 0)]$$

$$= 1 \times 2x \times \frac{1}{\cos x^2 \cdot \cos x^2} = 2x \sec^2 x^2.$$

Hence, $\frac{d}{dx}(\tan x^2) = 2x \sec^2 x^2.$

EXAMPLE 20 *Differentiate xe^x from the first principle.*

SOLUTION Let $y = xe^x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)\, e^{x+\delta x}$

$$\Rightarrow \quad \delta y = (x + \delta x)\, e^{x+\delta x} - xe^x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)\, e^{x+\delta x} - xe^x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)\, e^{x+\delta x} - xe^x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{xe^{x+\delta x} - xe^x + \delta x \cdot e^{x+\delta x}}{\delta x} = \lim_{\delta x \to 0} \left[xe^x \cdot \left(\frac{e^{\delta x} - 1}{\delta x} \right) + e^{x+\delta x} \right]$$

$$= xe^x \cdot \lim_{\delta x \to 0} \left(\frac{e^{\delta x} - 1}{\delta x} \right) + \lim_{\delta x \to 0} e^{x+\delta x}$$

$$= (xe^x \times 1) + e^x \qquad \left[\because \lim_{\delta x \to 0} \left(\frac{e^{\delta x} - 1}{\delta x} \right) = 1 \right]$$

$$= (xe^x + e^x) = (x + 1)\, e^x.$$

Hence, $\frac{d}{dx}(xe^x) = (x + 1)\, e^x.$

EXAMPLE 21 *Differentiate $x^2 \cos x$ from the first principle.*

SOLUTION Let $y = x^2 \cos x$.

Let δy be an increment in y, corresponding to an increment δx in x.

Then, $y + \delta y = (x + \delta x)^2 \cos(x + \delta x)$

$$\Rightarrow \quad \delta y = (x + \delta x)^2 \cos(x + \delta x) - x^2 \cos x$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 \cos(x + \delta x) - x^2 \cos x}{\delta x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x + \delta x)^2 \cos(x + \delta x) - x^2 \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(x^2 + \delta x^2 + 2x\delta x)\cos(x + \delta x) - x^2 \cos x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2[\cos(x + \delta x) - \cos x] + (\delta x)^2 \cos(x + \delta x) + 2x \cdot \delta x \cdot \cos(x + \delta x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{ \frac{x^2\left[-2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}\right] + (\delta x)^2 \cos(x + \delta x) + 2x \cdot \delta x \cdot \cos(x + \delta x)}{\delta x} \right\}$$

$$= \lim_{\delta x \to 0} \left[-x^2 \sin\left(x + \frac{\delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} + (\delta x)\cos(x + \delta x) + 2x\cos(x + \delta x) \right]$$

$$= -x^2 \cdot \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} + \lim_{\delta x \to 0} (\delta x)\cos(x + \delta x) + 2x \cdot \lim_{\delta x \to 0} \cos(x + \delta x)$$

$$= (-x^2 \times \sin x \times 1) + 0 + 2x\cos x = (-x^2 \sin x + 2x\cos x).$$

Hence, $\frac{d}{dx}(x^2 \cos x) = (-x^2 \sin x + 2x\cos x)$.

EXERCISE 28B

Find the derivative of each of the following from the first principle:

1. $(ax + b)$ **2.** $\left(ax^2 + \frac{b}{x}\right)$ **3.** $3x^2 + 2x - 5$ **4.** $x^3 - 2x^2 + x + 3$

5. x^8 **6.** $\frac{1}{x^3}$ **7.** $\frac{1}{x^5}$ **8.** $\sqrt{ax + b}$

9. $\sqrt{5x - 4}$ **10.** $\frac{1}{\sqrt{x + 2}}$ **11.** $\frac{1}{\sqrt{2x + 3}}$ **12.** $\frac{1}{\sqrt{6x - 5}}$

13. $\frac{1}{\sqrt{2 - 3x}}$ **14.** $\frac{2x + 3}{3x + 2}$ **15.** $\frac{5 - x}{5 + x}$ **16.** $\frac{x^2 + 1}{x}, x \neq 0$

17. $\sqrt{\cos 3x}$ **18.** $\sqrt{\sec x}$ **19.** $\tan^2 x$ **20.** $\sin(2x + 3)$

21. $\tan(3x + 1)$

ANSWERS (EXERCISE 28B)

1. a **2.** $\left(2ax - \frac{b}{x^2}\right)$ **3.** $6x + 2$ **4.** $3x^2 - 4x + 1$

5. $8x^7$ **6.** $\frac{-3}{x^4}$ **7.** $\frac{-5}{x^6}$ **8.** $\frac{a}{2\sqrt{ax+b}}$

9. $\frac{5}{2\sqrt{5x-4}}$ **10.** $\frac{-1}{2(x+2)^{3/2}}$ **11.** $\frac{-1}{(2x+3)^{3/2}}$ **12.** $\frac{-3}{(6x-5)^{3/2}}$

13. $\frac{3}{2(2-3x)^{3/2}}$ **14.** $\frac{-5}{(3x+2)^2}$ **15.** $\frac{-10}{(5+x)^2}$ **16.** $\left(1 - \frac{1}{x^2}\right)$

17. $\frac{-3}{2} \cdot \frac{\sin 3x}{\sqrt{\cos 3x}}$ **18.** $\frac{1}{2}\tan x \sqrt{\sec x}$ **19.** $2\tan x \sec^2 x$ **20.** $2\cos(2x+3)$

21. $3\sec^2(3x+1)$

HINTS TO SOME SELECTED QUESTIONS

7. $y = x^{-5} \Rightarrow (y+\delta y)(x+\delta x)^{-5}$

$\Rightarrow \frac{\delta y}{\delta x} = \frac{(x+\delta x)^{-5} - x^{-5}}{\delta x}$

$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{(x+\delta x)^{-5} - x^{-5}}{\delta x} = \lim_{\delta x \to 0} \frac{x^{-5}\left\{\left(1+\frac{\delta x}{x}\right)^{-5} - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{\left(1+\frac{\delta x}{x}\right)^{-5} - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{1 + (-5)\cdot\frac{\delta x}{x} + \frac{(-5)(-6)}{2}\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right\}}{\delta x}$$

$$= \frac{1}{x^5} \cdot \lim_{\delta x \to 0} \frac{\left\{(-5)\cdot\frac{\delta x}{x} + 15\cdot\left(\frac{\delta x}{x}\right)^2 + \ldots\right\}}{\delta x}$$

$$= \frac{1}{x^6} \cdot \lim_{\delta x \to 0} \left[(-5) + 15\left(\frac{\delta x}{x^2}\right) + \ldots\right]$$

$$= \frac{-5}{x^6}.$$

Hence, $\frac{d}{dx}(x^{-5}) = \frac{-5}{x^6}$.

9. $y=\sqrt{5x-4} \Rightarrow (y+\delta y)=\sqrt{5(x+\delta x)-4}$

$$\Rightarrow \frac{\delta y}{\delta x}=\frac{\sqrt{5(x+\delta x)-4}-\sqrt{5x-4}}{\delta x}.$$

$$\therefore \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{\sqrt{(5(x+\delta x)-4}-\sqrt{5x-4}}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{\{\sqrt{5(x+\delta x)-4}-\sqrt{5x-4}\}}{\delta x}\times\frac{\{\sqrt{5(x+\delta x)-4}+\sqrt{5x-4}\}}{\{\sqrt{5(x+\delta x)-4}+\sqrt{5x-4}\}}$$

$$=\lim_{\delta x\to 0}\frac{\{5(x+\delta x)-4\}-\{5x-4\}}{\delta x\{\sqrt{5(x+\delta x)-4}+\sqrt{5x-4}\}}$$

$$=\lim_{\delta x\to 0}\frac{5}{\{\sqrt{5(x+\delta x)-4}+\sqrt{5x-4}\}}=\frac{5}{2\sqrt{5x-4}}.$$

Hence, $\frac{d}{dx}(\sqrt{5x-4})=\frac{5}{2\sqrt{5x-4}}$.

10. $y=\frac{1}{\sqrt{x+2}} \Rightarrow (y+\delta y)=\frac{1}{\sqrt{x+\delta x+2}}$

$$\Rightarrow \frac{\delta y}{\delta x}=\left\{\frac{1}{\sqrt{x+\delta x+2}}-\frac{1}{\sqrt{x+2}}\right\}.\frac{1}{\delta x}$$

$$\therefore \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}$$

$$=\lim_{\delta x\to 0}\frac{1}{\delta x}\left\{\frac{1}{\sqrt{x+\delta x+2}}-\frac{1}{\sqrt{x+2}}\right\}$$

$$=\lim_{\delta x\to 0}\frac{1}{\delta x}.\frac{(\sqrt{x+2}-\sqrt{x+\delta x+2})(\sqrt{x+2}+\sqrt{x+\delta x+2})}{(\sqrt{x+\delta x+2})(\sqrt{x+2})(\sqrt{x+2}+\sqrt{x+\delta x+2})}$$

$$=\lim_{\delta x\to 0}\frac{\{(x+2)-(x+\delta x+2)\}}{\delta x(\sqrt{x+\delta x+2})(\sqrt{x+2})}.\frac{1}{(\sqrt{x+2}+\sqrt{x+\delta x+2})}$$

$$=\lim_{\delta x\to 0}\frac{-1}{(\sqrt{x+\delta x+2})(\sqrt{x+2})}.\frac{1}{(\sqrt{x+2}+\sqrt{x+\delta x+2})}$$

$$=\frac{-1}{2(x+2)^{3/2}}.$$

Hence, $\frac{d}{dx}\left(\frac{1}{\sqrt{x+2}}\right)=\frac{-1}{2(x+2)^{3/2}}$.

13. $y=\frac{1}{\sqrt{2-3x}} \Rightarrow (y+\delta y)=\frac{1}{\sqrt{2-3(x+\delta x)}}$

$$\Rightarrow \frac{\delta y}{\delta x}=\frac{1}{\delta x}.\left\{\frac{1}{\sqrt{2-3(x+\delta x)}}-\frac{1}{\sqrt{2-3x}}\right\}$$

$$\therefore \frac{dy}{dx}=\lim_{\delta x\to 0}\frac{\delta y}{\delta x}\lim_{\delta x\to 0}\frac{1}{\delta x}.\left\{\frac{1}{\sqrt{2-3(x+\delta x)}}-\frac{1}{\sqrt{2-3x}}\right\}$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \cdot \frac{\{\sqrt{2-3x} - \sqrt{2-3(x+\delta x)}\}}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}}{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}}$$

$$= \lim_{\delta x \to 0} \frac{[(2-3x) - \{2-3(x+\delta x)\}]}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{1}{\{\sqrt{2-3x} + \sqrt{2-3(x+\delta x)}\}} \cdot \frac{1}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{3}{\{\sqrt{2-3(x+\delta x)}\}\{\sqrt{2-3x}\}} \cdot \frac{1}{\{\sqrt{2-3x} + \sqrt{2-3x(x+\delta x)}\}}$$

$$= \frac{3}{2(2-3x)\sqrt{2-3x}} = \frac{3}{2(2-3x)^{3/2}}.$$

Hence, $\frac{d}{dx}\left(\frac{1}{\sqrt{2-3x}}\right) = \frac{3}{2(2-3x)^{3/2}}$.

14. $y = \frac{2x+3}{3x+2} \Rightarrow (y+\delta y) = \frac{2(x+\delta x)+3}{3(x+\delta x)+2}$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{1}{\delta x} \cdot \left\{\frac{2x+3+2\delta x}{3x+2+3\delta x} - \frac{2x+3}{3x+2}\right\}.$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{(2x+3+2\delta x)(3x+2) - (2x+3)(3x+2+3\delta x)}{(3x+2+3\delta x)(3x+2)\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\{(2x+3)(3x+2) - (2x+3)(3x+2)\} - 5\delta x}{(3x+2+3\delta x)(3x+2)\delta x}$$

$$= \lim_{\delta x \to 0} \frac{-5}{(3x+2+3\delta x)(3x+2)} = \frac{-5}{(3x+2)^2}.$$

Hence, $\frac{d}{dx}\left(\frac{2x+3}{3x+2}\right) = \frac{-5}{(3x+2)^2}$.

16. $\left(\frac{x^2+1}{x}\right) = \left(x + \frac{1}{x}\right)$.

18. $y = \sqrt{\sec x} \Rightarrow y + \delta y = \sqrt{\sec(x+\delta x)}$

$$\Rightarrow \delta y = \sqrt{\sec(x+\delta x)} - \sqrt{\sec x}.$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sqrt{\sec(x+\delta x)} - \sqrt{\sec x}}{\delta x} = \lim_{\delta x \to 0} \frac{\left\{\frac{1}{\sqrt{\cos(x+\delta x)}} - \frac{1}{\sqrt{\cos x}}\right)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left\{\frac{[\sqrt{\cos x} - \sqrt{\cos(x+\delta x)}]}{\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}} \cdot \frac{[\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}}{\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}} \cdot \frac{1}{\delta x}\right\}$$

$$= \lim_{\delta x \to 0} \left[\frac{[\cos x - \cos(x+\delta x)]}{\delta x[\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}} \cdot \frac{1}{[\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}]}\right]$$

$$= \lim_{\delta x \to 0} \frac{2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}}{\delta x \cdot [\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}] \cdot [\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}]}$$

$$= \lim_{\delta x \to 0} \sin\left(x + \frac{\delta x}{2}\right) \cdot \lim_{\delta x \to 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot$$

$$\lim_{\delta x \to 0} \frac{1}{\{\sqrt{\cos(x+\delta x)} \cdot \sqrt{\cos x}\}\{\sqrt{\cos x} + \sqrt{\cos(x+\delta x)}\}}$$

$$= \left[\sin x \times 1 \times \frac{1}{(\cos x) \cdot 2\sqrt{\cos x}}\right] = \frac{1}{2} \tan x \sqrt{\sec x}.$$

19. $y = \tan^2 x \Rightarrow y + \delta y = \tan^2(x + \delta x)$

$\Rightarrow \delta y = \tan^2(x + \delta x) - \tan^2 x$

$\Rightarrow \dfrac{\delta y}{\delta x} = \dfrac{\tan^2(x+\delta x) - \tan^2 x}{\delta x}.$

$\therefore \dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$$= \lim_{\delta x \to 0} \frac{\left\{\frac{\sin^2(x+\delta x)}{\cos^2(x+\delta x)} - \frac{\sin^2 x}{\cos^2 x}\right\}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\sin^2(x+\delta x)\cos^2 x - \sin^2 x \cos^2(x+\delta x)}{\delta x\,[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{[\sin(x+\delta x)\cos x + \sin x \cos(x+\delta x)][\sin(x+\delta x)\cos x - \sin x \cos(x+\delta x)]}{\delta x[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(x+\delta x+x) \cdot \sin(x+\delta x-x)}{\delta x\,[\cos^2(x+\delta x) \cdot \cos^2 x]} = \lim_{\delta x \to 0} \frac{\sin(2x+\delta x) \cdot \sin \delta x}{\delta x\,[\cos^2(x+\delta x) \cdot \cos^2 x]}$$

$$= \lim_{\delta x \to 0} \frac{\sin(2x+\delta x)}{\cos^2(x+\delta x) \cdot \cos^2 x} \cdot \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$= \left(\frac{\sin 2x}{\cos^4 x} \times 1\right) = \frac{2 \sin x \cos x}{\cos^4 x} = 2 \tan x \sec^2 x.$$

Some More Results on Differentiation

DERIVATIVE OF THE PRODUCT OF FUNCTIONS

THEOREM (Product rule) *If $f(x)$ and $g(x)$ are two differentiable functions then $f(x) \cdot g(x)$ is also differentiable, and*

$$\frac{d}{dx}\{f(x) \cdot g(x)\} = f(x) \cdot \frac{d}{dx}\{g(x)\} + g(x) \cdot \frac{d}{dx}\{f(x)\}.$$

PROOF Let $y = f(x) \cdot g(x)$ and $y + \delta y = f(x + \delta x) \cdot g(x + \delta x)$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x)\cdot g(x)}{\delta x}$.

$\therefore \dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x)\cdot g(x)}{\delta x}$

$= \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot g(x+\delta x) - f(x+\delta x)\cdot g(x) + f(x+\delta x)\cdot g(x) - f(x)\cdot g(x)}{\delta x}$

[adding and subtracting $f(x+\delta x)\cdot g(x)$ in num.]

$= \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x)\cdot\{g(x+\delta x) - g(x)\} + g(x)\cdot\{f(x+\delta x) - f(x)\}}{\delta x}$

$= \lim\limits_{\delta x \to 0} f(x+\delta x)\cdot\dfrac{\{g(x+\delta x) - g(x)\}}{\delta x} + \lim\limits_{\delta x \to 0} g(x)\cdot\left\{\dfrac{f(x+\delta x) - f(x)}{\delta x}\right\}$

$= \lim\limits_{\delta x \to 0} f(x+\delta x)\cdot \lim\limits_{\delta x \to 0} \dfrac{g(x+\delta x) - g(x)}{\delta x} + g(x)\cdot \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$

$= f(x)\cdot\dfrac{d}{dx}\{g(x)\} + g(x)\cdot\dfrac{d}{dx}\{f(x)\}.$

$\therefore \dfrac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\dfrac{d}{dx}\{g(x)\} + g(x)\cdot\dfrac{d}{dx}\{f(x)\}.$

Now, $f(x)$ and $g(x)$ being differentiable, it follows that $\dfrac{d}{dx}\{f(x)\}$ as well as $\dfrac{d}{dx}\{g(x)\}$ exists.

Consequently, $f(x)\cdot\dfrac{d}{dx}\{g(x)\} + g(x)\cdot\dfrac{d}{dx}\{f(x)\}$ exists.

So, from the above result, we conclude that $\dfrac{d}{dx}\{f(x)\cdot g(x)\}$ exists and therefore, $f(x)\cdot g(x)$ is differentiable.

REMARK The above result may be expressed as $\dfrac{d}{dx}[uv] = u\cdot\dfrac{dv}{dx} + v\cdot\dfrac{du}{dx}$.
It may be remembered as
derivative of the product of two functions
= [(1*st function*) × (*derivative of* 2*nd*)]
+ [(2*nd function*) × (*derivative of* 1*st*)].

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate:* (*i*) xe^x (*ii*) $x^2e^x\sin x$

SOLUTION (i) $\dfrac{d}{dx}(xe^x) = x\cdot\dfrac{d}{dx}(e^x) + e^x\cdot\dfrac{d}{dx}(x) = (xe^x + e^x\cdot 1) = e^x(x+1).$

$$\text{(ii)}\ \frac{d}{dx}(x^2e^x \sin x) = \frac{d}{dx}[(x^2e^x)\sin x]$$

$$= (x^2e^x)\cdot\frac{d}{dx}(\sin x) + \sin x\cdot\frac{d}{dx}(x^2e^x)$$

$$= x^2e^x\cos x + \sin x\cdot\left\{x^2\cdot\frac{d}{dx}(e^x) + e^x\cdot\frac{d}{dx}(x^2)\right\}$$

$$= x^2e^x\cos x + \sin x\cdot[x^2e^x + 2xe^x]$$

$$= x^2e^x\cos x + x^2e^x\sin x + 2xe^x\sin x$$

$$= xe^x[x\cos x + x\sin x + 2\sin x].$$

EXAMPLE 2 *Differentiate* $x^2\tan x$.

SOLUTION We have $\frac{d}{dx}(x^2\tan x)$

$$= \left\{x^2\cdot\frac{d}{dx}(\tan x) + \tan x\cdot\frac{d}{dx}(x^2)\right\}$$

$$= (x^2\sec^2 x + 2x\tan x).$$

EXAMPLE 3 *Differentiate* $(e^x\sin x + x^p\cos x)$.

SOLUTION We have $\frac{d}{dx}(e^x\sin x + x^p\cos x)$

$$= \frac{d}{dx}(e^x\sin x) + \frac{d}{dx}(x^p\cos x)$$

$$= \left\{e^x\cdot\frac{d}{dx}(\sin x) + \sin x\cdot\frac{d}{dx}(e^x)\right\} + \left\{x^p\cdot\frac{d}{dx}(\cos x) + \cos x\cdot\frac{d}{dx}(x^p)\right\}$$

$$= (e^x\cos x + e^x\sin x) + [x^p(-\sin x) + \cos x\cdot(px^{p-1})]$$

$$= e^x(\cos x + \sin x) - x^p\sin x + px^{p-1}\cos x$$

$$= e^x(\cos x + \sin x) - x^{p-1}(x\sin x - p\cos x).$$

EXAMPLE 4 *Differentiate* $e^x(x^3 + \sqrt{x})$.

SOLUTION $\frac{d}{dx}[e^x(x^3 + \sqrt{x})] = e^x\cdot\frac{d}{dx}(x^3 + \sqrt{x}) + (x^3 + \sqrt{x})\cdot\frac{d}{dx}(e^x)$

$$= e^x\left\{3x^2 + \frac{1}{2}x^{-1/2}\right\} + (x^3 + \sqrt{x})\cdot e^x$$

$$= e^x\left\{x^3 + 3x^2 + \frac{1}{2\sqrt{x}} + \sqrt{x}\right\}.$$

EXAMPLE 5 *Differentiate* $\left(\frac{e^x\cos x}{x^3}\right)$, *using the product rule.*

SOLUTION We have $\frac{d}{dx}\left\{\frac{e^x\cos x}{x^3}\right\} = \frac{d}{dx}[(e^x\cos x)\cdot x^{-3}]$

$$= e^x\cos x\cdot\frac{d}{dx}(x^{-3}) + x^{-3}\cdot\frac{d}{dx}(e^x\cos x)$$

$$= e^x \cos x \cdot (-3x^{-4}) + x^{-3} \cdot \left\{ e^x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(e^x) \right\}$$

$$= \frac{-3e^x \cos x}{x^4} + \frac{1}{x^3}[-e^x \sin x + e^x \cos x]$$

$$= \frac{-3e^x \cos x - xe^x \sin x + xe^x \cos x}{x^4} = \frac{e^x[(x-3)\cos x - x \sin x]}{x^4}.$$

EXAMPLE 6 *If u, v, w are differentiable functions of x, prove that*

$$\frac{d}{dx}(uvw) = (uv) \cdot \frac{dw}{dx} + (wu) \cdot \frac{dv}{dx} + (wv) \cdot \frac{du}{dx}.$$

SOLUTION We have $\frac{d}{dx}(uvw) = \frac{d}{dx}\{(uv)\, w\} = (uv) \cdot \frac{dw}{dx} + w \cdot \frac{d}{dx}(uv)$

$$= (uv) \cdot \frac{dw}{dx} + w \cdot \left\{ u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx} \right\}$$

$$= (uv) \cdot \frac{dw}{dx} + (wu) \cdot \frac{dv}{dx} + (wv) \cdot \frac{du}{dx}.$$

EXERCISE 28C

Differentiate:

1. $x^2 \sin x$ **2.** $e^x \cos x$ **3.** $e^x \cot x$

4. $x^n \cot x$ **5.** $x^3 \sec x$ **6.** $(x^2 + 3x + 1) \sin x$

7. $x^4 \tan x$ **8.** $(3x - 5)(4x^2 - 3 + e^x)$ **9.** $(x^2 - 4x + 5)(x^3 - 2)$

10. $(x^2 + 2x - 3)(x^2 + 7x + 5)$ **11.** $(\tan x + \sec x)(\cot x + \operatorname{cosec} x)$

12. $(x^3 \cos x - 2^x \tan x)$

ANSWERS (EXERCISE 28C)

1. $(x^2 \cos x + 2x \sin x)$ **2.** $e^x(\cos x - \sin x)$

3. $e^x (\cot x - \operatorname{cosec}^2 x)$ **4.** $x^{n-1}(n \cot x - x \operatorname{cosec}^2 x)$

5. $x^3 \sec x \tan x + 3x^2 \sec x$ **6.** $(x^2 + 3x + 1) \cos x + (2x + 3) \sin x$

7. $x^4 \sec^2 x + 4x^3 \tan x$ **8.** $(36x^2 - 40x + 3xe^x - 2e^x - 9)$

9. $5x^4 - 16x^3 + 15x^2 - 4x + 8$ **10.** $4x^3 + 27x^2 + 32x - 11$

11. $(\sec x - \operatorname{cosec} x)(\sec x + \tan x)(\operatorname{cosec} x + \cot x)$

12. $x^2(3 \cos x - x \sin x) - 2^x[(\log 2) \tan x + \sec^2 x]$

Derivative of the Quotient of two Functions

THEOREM (Quotient rule) *If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$ then $\dfrac{f(x)}{g(x)}$ is also differentiable, and*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

PROOF Let $y = \dfrac{f(x)}{g(x)}$ and $y + \delta y = \dfrac{f(x+\delta x)}{g(x+\delta x)}$.

Then, $\dfrac{\delta y}{\delta x} = \dfrac{1}{\delta x}\cdot\left\{\dfrac{f(x+\delta x)}{g(x+\delta x)} - \dfrac{f(x)}{g(x)}\right\}$.

$$\therefore \quad \frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\left\{\frac{f(x+\delta x)}{g(x+\delta x)} - \frac{f(x)}{g(x)}\right\}$$

$$= \lim_{\delta x\to 0}\frac{g(x)\cdot f(x+\delta x) - g(x+\delta x)\cdot f(x)}{\delta x\cdot g(x+\delta x)\cdot g(x)}$$

$$= \lim_{\delta x\to 0}\frac{g(x)\cdot f(x+\delta x) - g(x)\cdot f(x) + g(x)\cdot f(x) - g(x+\delta x)\cdot f(x)}{\delta x\cdot g(x+\delta x)\cdot g(x)}$$

[on subtracting and adding $g(x)\cdot f(x)$ in numerator]

$$= \left[\lim_{\delta x\to 0} g(x)\cdot\left\{\frac{f(x+\delta x)-f(x)}{\delta x}\right\} - \lim_{\delta x\to 0} f(x)\cdot\left\{\frac{g(x+\delta x)-g(x)}{\delta x}\right\}\right] \times \left[\lim_{\delta x\to 0}\frac{1}{g(x+\delta x)\cdot g(x)}\right]$$

$$= \left[g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}\right]\times\frac{1}{[g(x)]^2}$$

$$= \frac{g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

$$\therefore \quad \frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}}{[g(x)]^2}.$$

Now, each of $f(x)$ and $g(x)$ being differentiable, it follows that $\dfrac{d}{dx}\{f(x)\}$ and $\dfrac{d}{dx}\{g(x)\}$ both exist.

So, by the above result, it follows that $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\}$ exists, and hence $\dfrac{f(x)}{g(x)}$ is differentiable.

REMARK The above result may be expressed as $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\left\{v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}\right\}}{v^2}$, i.e.,

derivative of the quotient of two functions

$$=\frac{(\textit{denom.}\times\textit{derivative of num.})-(\textit{num.}\times\textit{derivative of denom.})}{(\textit{denominator})^2}$$

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate:*

(i) $\frac{e^x}{x}$ (ii) $\left(\frac{2x+3}{x^2-5}\right)$ (iii) $\frac{e^x}{(1+\sin x)}$

SOLUTION (i) $\frac{d}{dx}\left(\frac{e^x}{x}\right)=\frac{x\cdot\frac{d}{dx}(e^x)-e^x\cdot\frac{d}{dx}(x)}{x^2}=\frac{xe^x-e^x\cdot 1}{x^2}=\frac{e^x(x-1)}{x^2}$.

(ii) $\frac{d}{dx}\left(\frac{2x+3}{x^2-5}\right)=\frac{(x^2-5)\cdot\frac{d}{dx}(2x+3)-(2x+3)\cdot\frac{d}{dx}(x^2-5)}{(x^2-5)^2}$

$=\frac{(x^2-5)\cdot 2-(2x+3)\cdot 2x}{(x^2-5)^2}=\frac{-2(x^2+3x+5)}{(x^2-5)^2}$.

(iii) $\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right)=\frac{(1+\sin x)\cdot\frac{d}{dx}(e^x)-e^x\cdot\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$

$=\frac{(1+\sin x)\cdot e^x-e^x(\cos x)}{(1+\sin x)^2}=\frac{(1+\sin x-\cos x)\,e^x}{(1+\sin x)^2}$.

EXAMPLE 2 *Differentiate* $\left(\frac{x^2+5x-6}{4x^2-x+3}\right)$.

SOLUTION Using the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x^2+5x-6}{4x^2-x+3}\right)$$

$$=\frac{(4x^2-x+3)\cdot\frac{d}{dx}(x^2+5x-6)-(x^2+5x-6)\cdot\frac{d}{dx}(4x^2-x+3)}{(4x^2-x+3)^2}$$

$$=\frac{(4x^2-x+3)(2x+5)-(x^2+5x-6)(8x-1)}{(4x^2-x+3)^2}=\frac{(9+54x-21x^2)}{(4x^2-x+3)^2}.$$

EXAMPLE 3 *Differentiate* $\left(\frac{x^2\sin x}{1-x}\right)$.

SOLUTION By the quotient rule, we have

$$\frac{d}{dx}\left(\frac{x^2 \sin x}{1-x}\right) = \frac{(1-x)\cdot\frac{d}{dx}(x^2 \sin x) - x^2 \sin x \cdot \frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)\left\{x^2 \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x^2)\right\} - (x^2 \sin x)(-1)}{(1-x)^2}$$

$$= \frac{(1-x)[x^2 \cos x + 2x \sin x] + x^2 \sin x}{(1-x)^2}$$

$$= \frac{x^2(1-x)\cos x + (x \sin x)(2-x)}{(1-x)^2}.$$

EXAMPLE 4 *If* $y = \left\{\frac{1-\tan x}{1+\tan x}\right\}$, *show that* $\frac{dy}{dx} = \frac{-2}{(1+\sin 2x)}$.

SOLUTION By the quotient rule, we have

$$\frac{dy}{dx} = \frac{(1+\tan x)\cdot\frac{d}{dx}(1-\tan x) - (1-\tan x)\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{-2\sec^2 x}{(1+\tan x)^2} = \frac{-2}{(\cos^2 x)(1+\tan^2 x + 2\tan x)}$$

$$= \frac{-2}{(\cos^2 x)\left\{1 + \frac{\sin^2 x}{\cos^2 x} + \frac{2\sin x}{\cos x}\right\}} = \frac{-2}{(1+\sin 2x)}.$$

EXAMPLE 5 *Differentiate:*

(i) $\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$ *(ii)* $\left(\frac{\sec x - 1}{\sec x + 1}\right)$

SOLUTION (i) $\frac{d}{dx}\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$

$$= \frac{(\sin x - \cos x)\cdot\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\cdot\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-[(\sin x - \cos x)^2 + (\sin x + \cos x)^2]}{(\sin x - \cos x)^2}$$

$$= \frac{-2(\sin^2 x + \cos^2 x)}{(\sin^2 x + \cos^2 x - 2\sin x \cos x)} = \frac{-2}{(1-\sin 2x)}.$$

(ii) $\dfrac{d}{dx}\left(\dfrac{\sec x-1}{\sec x+1}\right)$

$$=\frac{(\sec x+1)\cdot\frac{d}{dx}(\sec x-1)-(\sec x-1)\cdot\frac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$=\frac{(\sec x+1)\sec x\tan x-(\sec x-1)\sec x\tan x}{(\sec x+1)^2}=\frac{2\sec x\tan x}{(\sec x+1)^2}.$$

EXERCISE 28D

Differentiate:

1. $\dfrac{2^x}{x}$ **2.** $\dfrac{\log x}{x}$ **3.** $\dfrac{e^x}{(1+x)}$ **4.** $\dfrac{e^x}{(1+x^2)}$

5. $\left(\dfrac{2x^2-4}{3x^2+7}\right)$ **6.** $\left(\dfrac{x^2+3x-1}{x+2}\right)$ **7.** $\dfrac{(x^2-1)}{(x^2+7x+1)}$ **8.** $\left(\dfrac{5x^2+6x+7}{2x^2+3x+4}\right)$

9. $\dfrac{x}{(a^2+x^2)}$ **10.** $\dfrac{x^4}{\sin x}$ **11.** $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$ **12.** $\dfrac{\cos x}{\log x}$

13. $\dfrac{2\cot x}{\sqrt{x}}$ **14.** $\dfrac{\sin x}{(1+\cos x)}$ **15.** $\left(\dfrac{1+\sin x}{1-\sin x}\right)$ **16.** $\left(\dfrac{1-\cos x}{1+\cos x}\right)$

17. $\left(\dfrac{\sin x-\cos x}{\sin x+\cos x}\right)$ **18.** $\left(\dfrac{\sec x-\tan x}{\sec x+\tan x}\right)$ **19.** $\left(\dfrac{e^x+\sin x}{1+\log x}\right)$

20. $\dfrac{e^x\sin x}{\sec x}$ **21.** $\dfrac{2^x\cot x}{\sqrt{x}}$ **22.** $\dfrac{e^x(x-1)}{(x+1)}$

23. $\dfrac{x\tan x}{(\sec x+\tan x)}$ **24.** $\left(\dfrac{ax^2+bx+c}{px^2+qx+r}\right)$ **25.** $\left(\dfrac{\sin x-x\cos x}{x\sin x+\cos x}\right)$

26. (i) $\cot x$ (ii) $\sec x$ [**Hint** Write $\cot x=\dfrac{\cos x}{\sin x}$ and $\sec x=\dfrac{1}{\cos x}$.]

ANSWERS (EXERCISE 28D)

1. $\dfrac{2^x(x\log 2-1)}{x^2}$ **2.** $\dfrac{(1-\log x)}{x^2}$ **3.** $\dfrac{xe^x}{(1+x)^2}$ **4.** $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$

5. $\dfrac{52x}{(3x^2+7)^2}$ **6.** $\dfrac{(x^2+4x+7)}{(x+2)^2}$ **7.** $\dfrac{(7x^2+4x+7)}{(x^2+7x+1)^2}$

8. $\dfrac{3(x^2+4x+1)}{(2x^2+3x+4)^2}$ **9.** $\dfrac{(a^2-x^2)}{(a^2+x^2)^2}$ **10.** $\dfrac{x^3(4\sin x-x\cos x)}{\sin^2 x}$

11. $\dfrac{\sqrt{a}}{\sqrt{x}\cdot(\sqrt{a}-\sqrt{x})^2}$ **12.** $\dfrac{-(x\sin x\log x+\cos x)}{x(\log x)^2}$

13. $\dfrac{-(2x\operatorname{cosec}^2 x+\cot x)}{x^{3/2}}$ **14.** $\dfrac{1}{(1+\cos x)}$ **15.** $\dfrac{2\cos x}{(1-\sin x)^2}$

16. $\dfrac{2\sin x}{(1+\cos x)^2}$ **17.** $\dfrac{2}{(1+\sin 2x)}$ **18.** $\dfrac{2\sec x(\tan x-\sec x)}{(\sec x+\tan x)}$

19. $\dfrac{x(e^x+\cos x)(1+\log x)-(e^x+\sin x)}{x(1+\log x)^2}$ **20.** $e^x[\cos 2x+\sin x\cos x]$

21. $\dfrac{2^x[-x\operatorname{cosec}^2 x+x(\log 2)\cot x-\frac{1}{2}\log x]}{x^{3/2}}$ **22.** $\dfrac{e^x(x^2+1)}{(x+1)^2}$

23. $\dfrac{x\sec x(\sec x-\tan x)+\tan x}{(\sec x+\tan x)}$ **24.** $\dfrac{(aq-bp)x^2+2(ar-pc)x+(br-cq)}{(px^2+qx+r)^2}$

25. $\dfrac{x^2}{(x\sin x+\cos x)^2}$ **26.** (i) $-\operatorname{cosec}^2 x$ (ii) $\sec x\ \tan x$

Derivative of a Function of a Function

CHAIN RULE If $y=f(t)$ and $t=g(x)$ then $\dfrac{dy}{dx}=\left(\dfrac{dy}{dt}\times\dfrac{dt}{dx}\right)$.

This rule may be extended further.

If $y=f(t), t=g(u)$ and $u=h(x)$ then $\dfrac{dy}{dx}=\left(\dfrac{dy}{dt}\times\dfrac{dt}{du}\times\dfrac{du}{dx}\right)$.

SOLVED EXAMPLES

EXAMPLE 1 *Differentiate* *(i)* $\sin x^3$ *(ii)* $\sin^3 x$ *(iii)* $e^{\sin x}$

SOLUTION (i) Let $y=\sin x^3$.

Put $x^3=t$, so that $y=\sin t$ and $t=x^3$.

$\therefore \quad \dfrac{dy}{dt}=\cos t$ and $\dfrac{dt}{dx}=3x^2$.

So, $\dfrac{dy}{dx}=\left(\dfrac{dy}{dt}\times\dfrac{dt}{dx}\right)$

$=3x^2\cos t=3x^2\cos x^3 \qquad [\because\ t=x^3]$.

Hence, $\dfrac{d}{dx}(\sin x^3)=3x^2\cos x^3$.

(ii) Let $y=\sin^3 x=(\sin x)^3$.

Put $\sin x=t$, so that $y=t^3$ and $t=\sin x$.

$$\therefore \quad \frac{dy}{dt} = 3t^2 \quad \text{and} \quad \frac{dt}{dx} = \cos x.$$

Hence, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$$= 3t^2 \cos x = 3\sin^2 x \cos x \qquad [\because \ t = \sin x].$$

(iii) Let $y = e^{\sin x}$.

Put $\sin x = t$, so that $y = e^t$ and $t = \sin x$

$$\therefore \quad \frac{dy}{dt} = e^t \quad \text{and} \quad \frac{dt}{dx} = \cos x.$$

Hence, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$$= e^t \cos x = e^{\sin x} \cdot \cos x \qquad [\because \ t = \sin x].$$

EXAMPLE 2 *If* $y = \dfrac{1}{\sqrt{a^2 - x^2}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Put $(a^2 - x^2) = t$, so that $y = \dfrac{1}{\sqrt{t}} = t^{-1/2}$ and $t = (a^2 - x^2)$.

$$\therefore \quad \frac{dy}{dt} = -\frac{1}{2}t^{-3/2} \quad \text{and} \quad \frac{dt}{dx} = -2x.$$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$$= \left(-\frac{1}{2}t^{-3/2}\right)(-2x) = xt^{-3/2} = x(a^2 - x^2)^{-3/2}.$$

EXAMPLE 3 *Differentiate:*

(i) $(ax + b)^m$ *(ii)* $(3x + 5)^6$ *(iii)* $\sqrt{ax^2 + 2bx + c}$

SOLUTION (i) Let $y = (ax + b)^m$.

Put $(ax + b) = t$, so that $y = t^m$ and $t = (ax + b)$.

$$\therefore \quad \frac{dy}{dt} = mt^{m-1} \quad \text{and} \quad \frac{dt}{dx} = a.$$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$$= amt^{m-1} = am\,(ax + b)^{m-1} \qquad [\because \ t = (ax + b)].$$

(ii) Let $y = (3x + 5)^6$.

Put $(3x + 5) = t$, so that $y = t^6$ and $t = (3x + 5)$.

$$\therefore \quad \frac{dy}{dt} = 6t^5 \quad \text{and} \quad \frac{dt}{dx} = 3.$$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right)$

$= 18t^5 = 18(3x + 5)^5 \qquad [\because\ t = (3x + 5)].$

(iii) Let $y = \sqrt{ax^2 + 2bx + c}$.

Put $(ax^2 + 2bx + c) = t$, so that $y = \sqrt{t}$ and $t = (ax^2 + 2bx + c)$.

$\therefore\ \dfrac{dy}{dt} = \dfrac{1}{2}t^{-1/2} = \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dt}{dx} = (2ax + 2b)$.

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{dt} \times \dfrac{dt}{dx}\right) = \dfrac{1}{2\sqrt{t}} \times 2(ax + b)$

$= \dfrac{(ax + b)}{\sqrt{t}} = \dfrac{(ax + b)}{\sqrt{ax^2 + 2bx + c}}.$

EXAMPLE 4 *Differentiate* $e^{\sqrt{\cot x}}$.

SOLUTION Let $y = e^{\sqrt{\cot x}}$. Put $\cot x = t$ and $\sqrt{\cot x} = \sqrt{t} = u$, so that

$y = e^u,\ u = \sqrt{t}$ and $t = \cot x.$

$\therefore\ \dfrac{dy}{du} = e^u,\ \dfrac{du}{dt} = \dfrac{1}{2}t^{-1/2} = \dfrac{1}{2\sqrt{t}}$ and $\dfrac{dt}{dx} = -\operatorname{cosec}^2 x.$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}\right) = -\dfrac{1}{2} \cdot \dfrac{\operatorname{cosec}^2 x}{\sqrt{t}}\, e^u$

$= \dfrac{-\operatorname{cosec}^2 x}{2\sqrt{t}} \cdot e^{\sqrt{t}} \qquad [\because\ u = \sqrt{t}]$

$= \dfrac{-\operatorname{cosec}^2 x}{2\sqrt{\cot x}} \cdot e^{\sqrt{\cot x}} \qquad [\because\ t = \cot x].$

EXAMPLE 5 *If* $y = \cos^2 x^2$, *find* $\dfrac{dy}{dx}$.

SOLUTION $y = (\cos x^2)^2$. Put $x^2 = t$ and $\cos x^2 = \cos t = u$, so that

$y = u^2,\ u = \cos t$ and $t = x^2.$

$\therefore\ \dfrac{dy}{du} = 2u,\ \dfrac{du}{dt} = -\sin t$ and $\dfrac{dt}{dx} = 2x.$

So, $\dfrac{dy}{dx} = \left(\dfrac{dy}{du} \times \dfrac{du}{dt} \times \dfrac{dt}{dx}\right)$

$= -4ux \sin t = -4x \sin t \cos t \qquad [\because\ u = \cos t]$

$= -4x \sin x^2 \cos x^2 = -2x \sin(2x^2) \qquad [\because\ t = x^2].$

EXAMPLE 6 *Differentiate* $\sqrt{\dfrac{1 - \tan x}{1 + \tan x}}$.

SOLUTION Let $y = \sqrt{\dfrac{1-\tan x}{1+\tan x}}$.

Putting $\dfrac{1-\tan x}{1+\tan x} = t$, we get $y = \sqrt{t}$ and $t = \dfrac{1-\tan x}{1+\tan x}$.

$$\therefore \quad \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}.$$

$$\text{And, } \frac{dt}{dx} = \frac{(1+\tan x)\cdot\frac{d}{dx}(1-\tan x) - (1-\tan x)\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(-\sec^2 x) - (1-\tan x)(\sec^2 x)}{(1+\tan x)^2} = \frac{-2\sec^2 x}{(1+\tan x)^2}.$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\times\frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}}\times\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^2}\times\frac{\sqrt{1+\tan x}}{\sqrt{1-\tan x}}$$

$$= \frac{-\sec^2 x}{(1+\tan x)^{3/2}(1-\tan x)^{1/2}}.$$

EXAMPLE 7 *If* $y = \sin(\sqrt{\sin x + \cos x})$, *find* $\dfrac{dy}{dx}$.

SOLUTION Putting $(\sin x + \cos x) = t$ and $\sqrt{(\sin x + \cos x)} = \sqrt{t} = u$, we get

$$y = \sin u, \quad u = \sqrt{t} \quad \text{and} \quad t = (\sin x + \cos x).$$

$$\therefore \quad \frac{dy}{du} = \cos u, \quad \frac{du}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$$

and $\dfrac{dt}{dx} = (\cos x - \sin x)$.

$$\text{So, } \frac{dy}{dx} = \left(\frac{dy}{du}\times\frac{du}{dt}\times\frac{dt}{dx}\right) = \frac{\cos u}{2\sqrt{t}}\cdot(\cos x - \sin x)$$

$$= \frac{\cos\sqrt{t}}{2\sqrt{t}}\cdot(\cos x - \sin x) \qquad [\because\ u = \sqrt{t}]$$

$$= \frac{\cos(\sqrt{\sin x + \cos x})(\cos x - \sin x)}{2\sqrt{\sin x + \cos x}} \quad [\because\ t = (\sin x + \cos x)].$$

EXAMPLE 8 *Differentiate* $\sin 2x \cos 3x$.

SOLUTION By using the product rule, we have

$$\frac{d}{dx}(\sin 2x \cos 3x) = (\sin 2x)\cdot\frac{d}{dx}(\cos 3x) + (\cos 3x)\cdot\frac{d}{dx}(\sin 2x)$$

$$= (\sin 2x)\cdot(-\sin 3x)\cdot 3 + (\cos 3x)\cdot(\cos 2x)\cdot 2$$

[using chain rule]

$$= (-3\sin 2x \sin 3x + 2\cos 3x \cos 2x).$$

EXAMPLE 9 *Differentiate* $e^{ax}\cos(bx+c)$.

SOLUTION Using the product rule, we have

$$\frac{d}{dx}\{e^{ax}\cos(bx+c)\}$$
$$= e^{ax}\cdot\frac{d}{dx}\{\cos(bx+c)\} + \cos(bx+c)\cdot\frac{d}{dx}(e^{ax})$$
$$= e^{ax}\{-\sin(bx+c)\}\cdot\frac{d}{dx}(bx+c) + \cos(bx+c)\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

[using the chain rule]

$$= -be^{ax}\sin(bx+c) + ae^{ax}\cos(bx+c)$$
$$= e^{ax}[a\cos(bx+c) - b\sin(bx+c)].$$

EXAMPLE 10 *Differentiate* (i) $\sqrt{\frac{1+x}{1-x}}$ (ii) $\frac{x}{\sqrt{1-x^2}}$

SOLUTION (i) Let $y = \sqrt{\frac{1+x}{1-x}}$.

Put $\frac{1+x}{1-x} = t$, so that $y = \sqrt{t}$ and $t = \frac{1+x}{1-x}$.

$\therefore \quad \frac{dy}{dt} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}$.

And, $\frac{dt}{dx} = \frac{(1-x)\cdot\frac{d}{dx}(1+x) - (1+x)\cdot\frac{d}{dx}(1-x)}{(1-x)^2}$

$$= \frac{(1-x)\cdot 1 - (1+x)(-1)}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$\therefore \quad \frac{dy}{dx} = \left(\frac{dy}{dt}\times\frac{dt}{dx}\right) = \frac{1}{2\sqrt{t}}\times\frac{2}{(1-x)^2}$

$$= \frac{1}{(1-x)^2\sqrt{\frac{1+x}{1-x}}} = \frac{\sqrt{1-x}}{(1-x)^2\sqrt{1+x}}$$
$$= \frac{1}{(1-x)^{3/2}(1+x)^{1/2}}.$$

(ii) $\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}\{\sqrt{1-x^2}\}}{(1-x^2)}$

$$= \frac{\sqrt{1-x^2}\cdot 1 - x\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)}{(1-x^2)}$$
$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{(1-x^2)+x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}}.$$

EXERCISE 28E

Differentiate the following with respect to x:

1. $\sin 4x$ **2.** $\cos 5x$ **3.** $\tan 3x$ **4.** $\cos x^3$

5. $\cot^2 x$ **6.** $\tan^3 x$ **7.** $\tan \sqrt{x}$ **8.** e^{x^4}

9. $e^{\cot x}$ **10.** $\sqrt{\sin x}$ **11.** $(5+7x)^6$ **12.** $(3-4x)^5$

13. $(3x^2 - x + 1)^4$ **14.** $(ax^2 + bx + c)^n$ **15.** $\dfrac{1}{(x^2 - x + 3)^3}$ **16.** $\sin^2(2x + 3)$

17. $\cos^2 x^3$ **18.** $\sqrt{\sin x^3}$ **19.** $\sqrt{x \sin x}$ **20.** $\sqrt{\cot \sqrt{x}}$

21. $\cos 3x \sin 5x$ **22.** $\sin x \sin 2x$ **23.** $\cos(\sin \sqrt{ax + b})$

24. $e^{2x} \sin 3x$ **25.** $e^{3x} \cos 2x$ **26.** $e^{-5x} \cot 4x$ **27.** $\cos(x^3 . e^x)$

28. $e^{(x \sin x + \cos x)}$ **29.** $\dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ **30.** $\dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$ **31.** $\sqrt{\dfrac{1 - x^2}{1 + x^2}}$

32. $\sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$ **33.** $\sqrt{\dfrac{1 + \sin x}{1 - \sin x}}$ **34.** $\sqrt{\dfrac{1 + e^x}{1 - e^x}}$ **35.** $\dfrac{e^{2x} + x^3}{\operatorname{cosec} 2x}$

Find $\dfrac{dy}{dx}$, *when*

36. $y = \sin(\sqrt{\sin x + \cos x})$ **37.** $y = e^x \log(\sin 2x)$

38. $y = \cos\left(\dfrac{1 - x^2}{1 + x^2}\right)$ **39.** $y = \sin\left(\dfrac{1 + x^2}{1 - x^2}\right)$

40. $y = \dfrac{\sin x + x^2}{\cot 2x}$

41. If $y = \dfrac{\cos x - \sin x}{\cos x + \sin x}$, show that $\dfrac{dy}{dx} + y^2 + 1 = 0$.

42. If $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$, show that $\dfrac{dy}{dx} = \sec^2\left(x + \dfrac{\pi}{4}\right)$.

43. If $y = \sqrt{\dfrac{1 - x}{1 + x}}$, prove that $(1 - x^2)\dfrac{dy}{dx} + y = 0$. **[CBSE 2004]**

44. If $y = \sqrt{\dfrac{\sec x - \tan x}{\sec x + \tan x}}$, show that $\dfrac{dy}{dx} = \sec x (\tan x + \sec x)$.

ANSWERS (EXERCISE 28E)

1. $4\cos 4x$ **2.** $-5\sin 5x$ **3.** $3\sec^2 3x$ **4.** $-3x^2 \sin x^3$

5. $-2\cot x\,\text{cosec}^2 x$ **6.** $3\tan^2 x\sec^2 x$ **7.** $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$ **8.** $4x^3(e^{x^4})$

9. $-(\text{cosec}^2\ x)(e^{\cot x})$ **10.** $\dfrac{\cos x}{2\sqrt{\sin x}}$ **11.** $42(5+7x)^5$ **12.** $-20(3-4x)^4$

13. $4(6x-1)(3x^2-x+1)^3$ **14.** $n(2ax+b)(ax^2+bx+c)^{n-1}$

15. $\dfrac{-3(2x-1)}{(x^2-x+3)^4}$ **16.** $2\sin(4x+6)$ **17.** $-3x^2\sin(2x^3)$ **18.** $\dfrac{3x^2\cos x^3}{2\sqrt{\sin x^3}}$

19. $\dfrac{(x\cos x+\sin x)}{2\sqrt{x\sin x}}$ **20.** $\dfrac{-\text{cosec}^2\sqrt{x}}{4\sqrt{x}\,.\sqrt{\cot\sqrt{x}}}$

21. $5\cos 3x\cos 5x-3\sin 3x\sin 5x$ **22.** $2\sin x\cos 2x+\cos x\sin 2x$

23. $\dfrac{-a}{2\sqrt{ax+b}}(\cos\sqrt{ax+b})\,.\,\sin(\sin\sqrt{ax+b})$ **24.** $e^{2x}(3\cos 3x+2\sin 3x)$

25. $e^{3x}(3\cos 2x-2\sin 2x)$ **26.** $-e^{-5x}(4\text{cosec}^2\ 4x+5\cot 4x)$

27. $-x^2\ e^x\ (x+3)\cdot\sin(x^3\ e^x)$ **28.** $(x\cos x).\,e^{(x\sin x+\cos x)}$

29. $\dfrac{-4}{(e^x-e^{-x})^2}$ **30.** $\dfrac{-8}{(e^{2x}-e^{-2x})^2}$ **31.** $\dfrac{-2x}{(1+x^2)^{3/2}(1-x^2)^{1/2}}$

32. $\dfrac{-2a^2x}{(a^2+x^2)^{3/2}(a^2-x^2)^{1/2}}$ **33.** $\dfrac{1}{(1-\sin x)}$ or $\sec x\,(\sec x+\tan x)$

34. $\dfrac{e^x}{(1-e^x)(1-e^{2x})^{1/2}}$ **35.** $2e^{2x}(\cos 2x+\sin 2x)+2x^3\cos 2x+3x^2\sin 2x$

36. $\dfrac{(\cos x-\sin x)\cos(\sqrt{\sin x+\cos x})}{2\sqrt{\sin x+\cos x}}$ **37.** $e^x[2\cot 2x+\log(\sin 2x)]$

38. $\dfrac{4x}{(1+x^2)^2}\cdot\sin\left(\dfrac{1-x^2}{1+x^2}\right)^2$ **39.** $\dfrac{4x}{(1-x^2)^2}\cdot\cos\left(\dfrac{1+x^2}{1-x^2}\right)$

40. $2(\sin x+x^2)\sec^2 2x+(\cos x+2x)\tan 2x$

HINTS TO SOME SELECTED QUESTIONS

35. Write $\dfrac{1}{\text{cosec}\,2x}=\sin 2x.$

42. $y=\dfrac{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}{\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x}=\dfrac{\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}}{\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4}}=\tan\left(x+\dfrac{\pi}{4}\right).$